Review and Clarification

In the last few lectures we have considered how an investor should allocate her wealth between different assets.

To solve the allocation problem we made some assumptions.

1. Utility maximization.
2. The investor likes expected return and dislikes variance.
3. Securities are infinitely divisible.
4. A frictionless financial market (borrow and sell at the riskfree rate, and costless to short-sell).
5. The investor takes prices as given.
6. The investor knows the expected return vector and covariance matrix of all the securities they can invest in.
Review and Clarification

- What do you think of these assumptions?
- Are they realistic?
- Are they too strong?

Given our assumptions what kind of portfolio does an investor choose?

- The investor chooses a mean variance efficient portfolio.
- A portfolio that for a given level of standard deviation has the highest expected return relative to all possible portfolios.
Thinking about Risk

An investor allocates her money between the tangency portfolio and the riskfree asset.

- The efficient frontier is the CAL line between the riskfree security and the tangency portfolio.

How should we measure risk in this framework?

- For the portfolio you own?
- For a portfolio you don’t own?
- For the security level?
The Tangency Portfolio

The tangency portfolio has the highest Sharpe ratio for a portfolio of just risky assets:

\[ SR = \frac{E(r_T) - r_f}{\sigma(r_T)} \]

The marginal Sharpe ratio condition of the tangency portfolio:

\[ \frac{E(r_i) - r_f}{\text{cov}(r_i, r_T)} = \frac{E(r_j) - r_f}{\text{cov}(r_j, r_T)} \quad \text{for all } i \text{ and } j \]

Implications:

- We find the tangency portfolio by picking the weights so the preceding condition holds.
- The condition holds for all individual securities and for all portfolios (as i or j). Why?

Since risk premium to covariance ratio equality holds for all assets, it must also hold for itself.

\[ \frac{E(r_i) - r_f}{\text{cov}(r_i, r_T)} = \frac{E(r_T) - r_f}{\text{cov}(r_T, r_T)} \]

But remember that \( \text{cov}(r_T, r_T) = \sigma^2(r_T) \). Therefore,

\[ \frac{E(r_i) - r_f}{\text{cov}(r_i, r_T)} = \frac{E(r_T) - r_f}{\sigma^2(r_T)} \]

We can use the above expression to derive a relation between an asset’s expected return and risk.
The Tangency Portfolio

The relation between expected return and risk with the tangency portfolio is typically written as,

\[ E(r_i) = r_f + \beta_{iT} [E(r_T) - r_f], \text{ where } \beta_{iT} = \frac{\text{cov}(r_i, r_T)}{\sigma^2(r_T)} \]

Differences in expected returns are explained by differences in \( \beta_{iT} \).

The relation between expected return and \( \beta_{iT} \) is linear.

- The riskfree rate is the y-intercept.
- The risk premium of the tangency portfolio is the slope.
Can We Use This?

Mean variance mathematics implies that,
\[ E(r_i) = r_f + \beta_{iT} [E(r_T) - r_f], \]
where \( \beta_{iT} = \frac{\text{cov}(r_i, r_T)}{\sigma^2(r_T)} \)

- What’s great about this relation?
- Why can’t we use this practically to estimate expected return or risk?
- What is missing or lacking?
We Need the Weights

How do we identify the tangency portfolio (i.e., find the weights)?

◆ If we make one more assumption, then economic theory will identify the tangency portfolio for us.

◆ When I say identify, I mean economic theory will tell us what the weights of the tangency portfolio should be.

Identifying the weights of the tangency portfolio:

◆ So far we have discussed the relation between risk and return that an individual investor faces.

◆ However, inorder to identify the tangency portfolio, we need to examine how prices are set in the capital market.

Assumption: Investors all make the same forecasts of expected returns, variances, and covariances.

◆ This is often called complete agreement.

◆ Very strong assumption that is almost certainly not true.

◆ Why make an assumption that we know doesn’t hold? Isn’t that silly?

Implication: M = T:

◆ In equilibrium, prices must be set so that the market portfolio (M) is the tangency portfolio (T).
The Market Portfolio

The portfolio of all risky assets where the assets are held in proportion to their market value. The return on such a portfolio is,

$$r_M = \sum_{i}^{n} v_i r_i$$

where

$$v_i = \frac{\text{total dollar value of asset } i}{\text{total dollar value of all assets}}$$

Market Equilibrium

Complete agreement and $M = T$:

- Given the assumption what can we say about investors portfolio choices?
- How are their allocations different from each other?
- What is the commonality across different investors holdings?
- Why does this assumption imply that $M=T$?
Finally, we have identified the tangency portfolio (after we made a few assumptions).

Prices must be set so that the market portfolio (M) is the tangency portfolio.

- The market portfolio contains all assets we can invest in: stocks, bonds, real-estate, human capital, etc.
- The good thing is we know the weights of the market portfolio:
  - The weight on any security is just the value of the security today divided by the total value of all securities.

The Capital Market Line (CML)

- The riskfree rate
- The Market Portfolio
- \( E(r) \)
- \( \sigma(r) \)
The CAPM

Since the market portfolio (M) is the tangency portfolio, mean variance mathematics implies the following:

$$E(r_i) = r_f + \beta_{iM}[E(r_M) - r_f], \text{ where } \beta_{iM} = \frac{\text{cov}(r_i, r_M)}{\sigma^2(r_M)}$$

The above equation is the Capital Asset Pricing Model.

The CAPM is a model where prices are set so that the market portfolio is the tangency portfolio.

In the CAPM, $\beta_{iM}$ is the measure of risk.

- There is a linear relation between expected return and $\beta_{iM}$.
- Differences in expected returns are due to differences in beta; no other variables matter.

The Security Market Line (SML)

$$E(r_i) = r_f + \beta_{iM}(E(r_M) - r_f)$$
Some Examples

Q: Suppose the risk free rate is 5%, and the expected return on the market portfolio is 11%, the standard deviation 20%, and the covariance of IBM with the market is 0.05. What is IBM’s beta and expected return?

\[ \beta_{ibmM} = \frac{\text{cov}(r_{ibm}, r_M)}{\sigma^2(r_M)} = \frac{0.05}{0.20^2} = 1.25 \]

\[ E(r_{ibm}) = r_f + \beta_{ibmM}[E(r_M) - r_f] = 0.05 + 1.25(0.11 - 0.05) = 12.5\% \]

The CAPM & Multifactor Models 20

Some Examples

Q: Suppose the risk free rate is 4%, the expected return on the market portfolio is 12%, and its standard deviation is 20%. Stock Z has a standard deviation of 50%, but is uncorrelated with the market. What is Z’s beta and expected return?

\[ \beta_{zM} = \frac{\text{cov}(r_z, r_M)}{\sigma^2(r_M)} = \frac{0.00}{0.20^2} = 0 \]

\[ E(r_z) = r_f + \beta_{zM}[E(r_M) - r_f] = 0.04 + 0(0.12 - 0.04) = 4\% \]

Thus even though Z has a large standard deviation its expected return is the riskfree rate.

The CAPM & Multifactor Models 21
Some Examples

Q: You invest 20% percent of your money in T-bills, and 80% percent in the market portfolio. What is the beta of your portfolio?

**Statistical Fact:** The beta of a portfolio is the weighted sum of the component assets’ betas.

\[
\beta_{pM} = \sum_{i=1}^{n} w_i \beta_{iM}
\]

\[
\beta_{pM} = 0.2 \beta_{fM} + 0.8 \beta_{MM}
\]

\[
= 0.2 \frac{\text{cov}(r_f, r_M)}{\sigma^2(r_M)} + 0.8 \frac{\text{cov}(r_M, r_M)}{\sigma^2(r_M)}
\]

\[
= 0 + 0.8 \frac{\sigma^2(r_M)}{\sigma^2(r_M)} = 0.8
\]

CAPM Intuition

High beta stocks are risky, and therefore investors will only buy them if they offer higher returns on average.

Why are high beta stocks risky?

- High beta stocks tend to have good returns when the economy and the market are doing well.
- High beta stocks tend to have poor returns when the economy is doing bad.
- If something is risky, then it is undesirable. You must be compensated to hold it. What is undesirable about high beta stocks?
Other Models

The CAPM is a model.

- No model is perfect.
- We will find that the CAPM has some significant weaknesses.

Other Models

- We need to consider other models if the CAPM is an empirical failure.
- All our other models will build on the economic theory and institution we used for the CAPM.

Factors and Priced Factors

Factor: A variable that explains why a group of stocks have returns that tend to move together.

- Put another way: a variable that helps explain common components of the variance of security returns.

A priced factor is a variable which helps explain the expected returns of assets.

- Put another way: a variable that helps explain common components of the variance of security returns and that helps explain expected returns.

- We usually think of a priced factor in the context of a rational asset pricing model (a risk factor).
A Single Factor Model: The CAPM

Q: What is the only priced factor in the CAPM?
A: The excess return on the market is the only priced factor in the CAPM.

\[ E(r_{it}) - r_{ft} = \beta_i M [E(r_{Mt}) - r_{ft}] \]

- Stocks have high expected returns if they have high sensitivity (\( \beta \)) with the factor.
- However, maybe one factor is not enough?

Are There Factors?

Q: Are there factors in security returns?
Everyone believes there are factors in security returns.

Controversy: Can we identify rational factors that explain the cross-section of expected returns.

- Every one agrees that the excess return on the market portfolio is a factor
- Debate: is that enough or are there other factors?
Types of Factors

Factors can be tradeable portfolios.

- They must be zero cost portfolios (weights add up to zero).
- For example, in the CAPM the portfolio is 100% in the market portfolio and -100% in the riskfree rate ($r_{Mt} - r_{ft}$).

Factors can also be non-tradeable variables such as macro-economic variables.

- Maybe GDP growth.
- Maybe unexpected inflation.

We will focus on the case where the factors are zero cost portfolios.

Detour: Zero Cost Portfolios

A zero cost portfolio is a portfolio where the weights add up to zero.

Usually we put zero cost portfolios in units such that the positive weights add up to 100% and the negative weights add up to -100%.

- Example: You go long IBM 100% and short T-bills 100%.
  - The return: $r_{ibm} - r_{f}$.
- Example: You go 100% long in IBM and short 100% in GE.
  - The return: $r_{ibm} - r_{ge}$.
Detour: Zero Cost Portfolios

Zero cost portfolios are also called **self financing portfolios**.

- If the market is frictionless then a zero-cost portfolio is free.
- It requires no capital outlay.

\[ r_i - r_j \] is called a **zero cost portfolio return**, **excess return**, or **a differential return**.

Multifactor Models

In general multifactor models will look like the following:

\[
E(r_p) - r_f = \sum_{j=1}^{K} \beta_{pj} E(F_j)
\]

In a multifactor model, the expected excess return of a security or portfolio is due entirely to factor loadings and factor risk premia.

- **Factor loadings**: \( \beta_{p1}, \beta_{p2}, \ldots, \beta_{pK} \)
- **Factor risk premia**: \( E(F_1), E(F_2), \ldots, E(F_K) \)

Note: Multifactor models are often referred to as Arbitrage Pricing Theory (APT) models.
Some Practical Issues

Suppose you believe that there are two priced risk factors:

- The excess return on the market portfolio.
- A crime factor.

What would your model look like?

How about the following?

\[
E(r_i) - r_f = \beta_iM[E(r_M) - r_f] + \beta_iC[E(\text{crime rate})]
\]

No, the factors should be zero cost portfolios.

Solution: make the crime factor a zero cost portfolio.

We can just form a portfolio of securities that benefit from high crime (gun companies, etc). Call the portfolio G.

Also form a portfolio of stocks that are hurt by high crime (jewelry stores, etc). Call the portfolio J.

The crime factor as a zero cost portfolio is,\n
\[ r_g - r_j \]

and the multifactor model is,\n
\[
E(r_i) - r_f = \beta_iM[E(r_M) - r_f] + \beta_iC(r_g - r_j)
\]
Thinking about Risk Factors

A priced or risk factor is a variable which helps explain the expected returns of assets.

- Risk factors are linked to investor happiness (marginal utility). That is why they affect expected returns.
- The excess return on the market will always be a risk factor in all our potential models.
- If the market excess return is a risk factor, what does this imply about any additional risk factors?

An Example: A Crime Factor

Investors don’t like crime; it makes them less happy.

Suppose there is a crime factor in stock returns.

- If crime goes up, then companies that produce alarms, guns, locks and security guards will see their returns go up.
- Returns go down for luxury car and jewelry companies.
- A stock that pays off more money when crime is high is desirable since it gives you money when you are less happy.
- A stock that has low returns during high crime is undesirable; you get little money when you are already unhappy.
- The crime factor will have low expected returns.
Some Final Points

Factors reflect common movement in security returns.

- Priced factors should explain expected returns for individual securities and portfolios.
- Figuring out the correct priced factors is difficult, and there is no general consensus.
- Risk and reward usually go together. If you want high returns on average buy things that do well in good times and poorly in bad times.