APPLICATION OF THE FINITE ELEMENT METHOD TO HEAT CONDUCTION ANALYSIS *

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A variational principle is applied to the transient heat conduction analysis of complex solids of arbitrary shape with temperature and heat flux boundary conditions. The finite element discretization technique is used to reduce the continuous spatial solution into a finite number of time-dependent unknowns. The continuum is divided into subregions (elements) in which the temperature field variable is approximated by Rayleigh-Ritz polynomial expansions in terms of the values of the temperatures at prescribed boundary points of the subregions. These temperatures at the node points act as generalized coordinates of the system and, being common to adjacent subregions, enable appropriate continuity requirements to be satisfied over the entire continuum.

The variational principle yields Euler equations of the Lagrangian form which result in the development of an equivalent set of first order, ordinary differential equations in terms of the nodal temperatures. A unique method of numerical solution of these equations is introduced which is stable and requires a minimum of computer effort. Elements of various shapes and their associated temperature fields are discussed for one, two and three-dimensional bodies. The method is developed in detail for two-dimensional bodies which are idealized by systems of triangular elements. The development of a digital computer program is discussed and several examples are given to illustrate the validity and practicality of the method.

1. INTRODUCTION

Several approximate methods of solution to heat conduction problems, which are based on variational principles, have been introduced in the past several years [1-4]. The generality of the variational approach with respect to arbitrary boundary conditions and material property variations has allowed for the solution of many complex problems. In this investigation, a variational principle is used in conjunction with the finite element idealization. This results in a powerful solution technique for the determination of the temperature distribution within complex bodies of arbitrary geometry.

In the finite element approximation of solids, the continuous body is replaced by a system of elements. In the case of heat conduction an approximate solution within each element for the temperature field is assumed and heat flux equilibrium equations are developed at a discrete number of points within the finite element system. For the case of steady-state heat flow the technique is completely described by Zienkiewicz [5]. The purpose of this paper is to extend the technique to the transient heat conduction problem.

The advantages of the finite element method, as compared to other numerical approaches, are numerous. The method is completely general with respect to geometry and material properties. Complex bodies composed of many different anisotropic materials are easily represented. Temperature or heat flux boundary conditions may be specified at any point within the finite element system. Mathematically, it can be shown that the method converges to the exact solution as the number of elements is increased; therefore, any desired degree of accuracy may be obtained. In addition, for the steady-state condition the finite element approach generates heat flow equilibrium equations which produce a symmetric, positive-definite matrix which may be placed in a band form and solved with a minimum of computer storage and time. Also, for the transient problem, a step-by-step procedure is introduced which takes advantage of the special characteristics of the matrices.
2. THE VARIATIONAL EXPRESSION

Adopting a similar notation to that of Gurtin [6] the functional:

$$\Omega_t(\theta) = \frac{1}{2} \int_V \left\{ \rho c \theta \cdot \theta + \rho \cdot \dot{\theta} + k_{ij} \cdot \nabla \theta \right\} dV - \int_S \left\{ Q \rho \theta \cdot \theta \right\} (x, t) dS$$

is introduced over the volume $V$ and the boundary $S$ of a continuum, where $\theta$ is the temperature field as a function of space and time, $k_{ij}$ is the thermal conductivity tensor which may be space dependent, $\rho(x)$ is the mass density and $c(x)$ is the heat capacity per unit mass. The rate at which heat is generated per unit mass is denoted by the function $p(x, t)$. Initial spatial temperature distribution is represented by $\theta_0(x)$ and

$$\dot{Q}_t(x, t) = \int_0^t Q_t(x, \tau) d\tau$$

represents prescribed heat flux conditions on the boundary $S$. The convolution symbol is defined as in ref. [6] by the expression:

$$\{u \ast v\} (x, t) = \int_0^t u(x, t - \tau) v(x, \tau) d\tau$$

Let $\Theta$ be the set of all functions continuous in the body and satisfying prescribed temperature boundary conditions on $S$. Then, by taking the first variation of eq. (1) and applying the divergence theorem, it is seen that $\delta \Omega_t = 0$ over $\Theta$ ($0 \leq t < \infty$) at a particular function $\theta$ if and only if $\theta$ is a solution of the Euler equation:

$$(k_{ij} \cdot \theta, j) - \rho c \theta + \rho c \theta_0 + \rho \cdot \dot{\theta} = 0$$

with the natural boundary conditions:

$$k_{ij} \cdot \theta, j - Q_t = 0.$$  

Through the use of the simple identity:

$$\{\rho c \cdot \theta\} (x, t) = \int_0^t \rho(x) c(x) \dot{\theta}(x, \tau) d\tau$$

$$= \rho(x) c(x) [\theta(x, t) - \theta_0(x)] .$$

Eq. (4) may be transformed into the more usual form of the transient heat conduction equation:

$$(k_{ij} \cdot \theta, j) - \rho c \cdot \dot{\theta} - \rho \cdot \dot{p} = 0 .$$

Hence the use of eq. (1) as the generating functional is verified.

3. THE FINITE ELEMENT IDEALIZATION

For the purpose of expressing the generating functional in terms of a finite number of unknowns, the body is subdivided into a finite number of regions. A typical representation of an axisymmetric body by a system of triangular regions is shown in fig. 1.

The solution for the temperature distribution within the body is now determined by the standard Rayleigh-Ritz procedure in which the generalized coordinates are selected as the temperatures at the nodal points of the finite element idealization. An admissible temperature field is one in which the function is continuous throughout the discretized system. It has been shown that
continuity of derivatives of the field between elements is not a requirement of the method [5]. The form of the assumed temperature field within a region (element) will depend on the specific type of element used. For a typical element \( m \) the assumed field is written in the following general form:

\[
\theta_m(\vec{x}, t) = \alpha_j(t) \phi_j(\vec{x}) , \quad j = 1, J ,
\]

where \( J \) is the number of element nodes (e.g. \( J \) equals three for a two-dimensional triangular element). The spatial functions \( \phi_j \) must be selected in such a manner that the temperature field is compatible between elements. The \( \alpha_j \) are related to the temperatures at the element nodes and are calculated by evaluating eq. (8) at these nodes. Therefore, the temperature at any point within the element may be expressed in terms of nodal point temperatures by the following matrix equation:

\[
\theta_m(\vec{x}, t) = \langle \phi_m(\vec{x}) \rangle \{ \theta(t) \} ,
\]

where \( \{ \theta(t) \} \) is a column vector of nodal point temperatures for the complete finite element system and the coefficients \( \langle \phi_m(\vec{x}) \rangle \) represent the spatial approximation. Most of the coefficients are zero since the temperature field within an element is completely defined by the nodal point temperatures in or adjacent to the element.

Differentiation of eq. (9) with respect to the spatial coordinates yields a column vector of temperature gradients which may be written symbolically as:

\[
\{ \partial_{\vec{x}, i} \theta_m(\vec{x}, t) \} = [a_m(\vec{x})] \{ \theta(t) \} .
\]

The generating functional, eq. (1), for the finite element system may be cast in the form of a summation over all elements:

\[
\Omega_t = \frac{1}{2} \{ \theta(t) \}^T \{ \theta(t) \} + \frac{1}{2} \{ \theta(t) \}^T \{ K \} \{ \theta(t) \} - \{ \theta(t) \}^T \{ C \} \{ \theta(0) \} - \{ \theta(0) \}^T \{ Q(t) \} ,
\]

where

\[
[C] = \sum_{m=1}^{M} [C^m] , \quad [K] = \sum_{m=1}^{M} [K^m] ,
\]

\[
\{ Q(t) \} = \sum_{m=1}^{M} \{ Q^m(t) \} ,
\]

and where \([C]\) is defined as the heat capacity matrix equal to the sum of the element heat capacity matrices:

\[
[C^m] = \int_{V_m} \rho_m(\vec{x}) C_m(\vec{x}) \langle \phi_m(\vec{x}) \rangle^T \langle \phi_m(\vec{x}) \rangle \, dV_m .
\]

(14a)

\([K]\) is defined as the conductivity matrix and is equal to the sum of the element conductivity matrices:

\[
[K^m] = \int_{V_m} [a_m(\vec{x})]^T [k^m(\vec{x})] [a_m(\vec{x})] \, dV_m ,
\]

(14b)

and \( \{ Q \} \) is defined as the thermal force vector, equal to the sum of the element thermal force vectors:

\[
\{ Q^m(t) \} = \int_{V_m} \rho_m(\vec{x}) \rho(\vec{x}, t) \langle \phi_m(\vec{x}) \rangle^T \, dV_m + \int_{S_m} Q^m_i(\vec{x}, t) n_i \langle \phi_m(\vec{x}) \rangle^T \, dS_m .
\]

(14c)

The first variation of eq. (12) yields:

\[
[C] \{ \theta(t) \} + [K] \{ \dot{\theta}(t) \} = [C] \{ \theta(0) \} + \{ Q(t) \} ,
\]

(15)

a set of linear equations to be solved for the nodal point temperatures of the finite element representation as a function of time.

4. SURFACE HEAT TRANSFER

The rate of heat flux across a boundary layer at the surface of a body is:

\[
q = h * (\theta_e - \theta) ,
\]

(16)

where \( \theta_e \) is the known temperature outside the layer, \( \theta \) is the unknown temperature at the surface of the body, and \( h \) is the heat-transfer coefficient for the layer. This surface heat-transfer
effect is considered by subtracting the following surface integral from eq. (1):

\[
\int_S \{ k \ast (\theta_e - \frac{1}{2} \theta) \ast \theta \} (E, t) \, dS . \tag{17}
\]

For a finite element idealization the surface temperature field for a typical boundary element \( m \) is expressed in terms of the nodal temperature by an approximate relationship:

\[
\theta_m(x, t) = \langle b_m(x) \rangle \{ \theta (t) \} . \tag{18}
\]

The approximate surface temperature field must be compatible with the approximate volume temperature field, eq. (9).

The inclusion of eq. (17) in the variational expression leads to the following form of the element conductivity matrix, eq. (14b), and the element thermal force matrix, eq. (14c):

\[
[K^m] = \int_{V_m} [a_m]^T [k^m] [a_m] \, dV_m \\
+ \int_{S_m} \rho_m [d_m]^T \{ d_m \} \, dS_m , \tag{19a}
\]

\[
\{ \rho^m \} = \int_{V_m} \rho_m [d_m]^T \{ b_m \} \, dV_m \\
+ \int_{S_m} \rho_m [d_m]^T \{ b_m \} \, dS_m + \int_{S_m} \theta_e \ast h_m \{ d_m \} \, dS_m . \tag{19b}
\]

The added surface integrals are required only for elements which are subjected to this heat-transfer effect.

5. TEMPERATURE BOUNDARY CONDITIONS

A temperature distribution as a function of time may be specified along a boundary of the body by specifying the temperature at a discrete number of nodal points along the boundary. This is accomplished by rewriting eq. (15) in a matrix partitioned form:

\[
\begin{pmatrix}
C_{aa} & C_{ab} \\
C_{ba} & C_{bb}
\end{pmatrix}
\begin{pmatrix}
\theta_a \\
\theta_b
\end{pmatrix} +
\begin{pmatrix}
K_{aa} & K_{ab} \\
K_{ba} & K_{bb}
\end{pmatrix}
\begin{pmatrix}
\theta_a \\
\theta_b
\end{pmatrix} =
\begin{pmatrix}
\tilde{Q}_a \\
\tilde{Q}_b
\end{pmatrix} +
\begin{pmatrix}
Q_{a0} \\
Q_{b0}
\end{pmatrix} . \tag{20a}
\]

\[
\{ \tilde{Q}_a \} = \{ Q_a \} - [C_{ab}] \{ \theta_b \} - [K_{ab}] \ast \{ \theta_b \} \\
+ [C_{ab}] \{ \theta_b (0) \} . \tag{20b}
\]

The effective thermal force matrix, \( \{ \tilde{Q}_a \} \), is completely defined and the solution of eq. (21a) for the unknown temperatures may be accomplished by standard techniques. If desired, the unknown thermal forces, \( \{ Q_{b0} \} \), which are associated with the specified temperature points, are then calculated directly from the second partitioned equation, eq. (20b).

6. STEP-BY-STEP SOLUTION PROCEDURE

The nodal point temperatures for a finite element system, including temperature and heat flux boundary conditions, must satisfy eq. (21a). The solution may be obtained, as suggested by Biot [2], by the standard mode superposition method in which the initial step involves the evaluation of the thermal modes and characteristic roots. Since eq. (21a) may contain several hundred unknowns for a typical finite element system, the mode superposition approach is not practical because of the tremendous number of numerical operations required to solve a characteristic value problem of this size.

In order to develop the step-by-step technique eq. (21a) is rewritten to reflect the solution at time \( t \) in terms of pseudo-initial values at time \( t - \Delta t \):

\[
[C] \{ \theta (t) \} + [K] \ast \{ \theta (t) \} = [C] \{ \theta (t - \Delta t) \} + \{ Q (t) \} . \tag{22}
\]

Over the interval \(( t, t - \Delta t )\) the nodal point temperature vector will be approximated by:

\[
\{ \theta (\xi) \} = \{ F_1 \} + [ \xi - (t - \Delta t) ] \{ F_2 \} . \tag{23}
\]

The calculation of the thermal force vector requires the time integration of the heat flux and convective boundary conditions as well as the
volumetric heat generated in the body. These functional values are assumed to vary linearly over the same time interval \((t, t - \Delta t)\).

Evaluating eq. (23) at the end points of the interval, thus insuring continuity of the temperature field with respect to the time variable, the expression for the nodal point temperature vector becomes:

\[
\{\theta(t)\} = \left(\frac{t - \xi}{\Delta t}\right) \{\theta\}_{t-\Delta t} + \frac{\xi}{\Delta t} \{\theta\}_t .
\]  

Substituting these expressions into eq. (22) results in the following matrix equation for the finite element system:

\[
\left(\left[ C + \frac{1}{2} \Delta t \left[ K \right] \right]\right) \{\theta\}_t = \left(\left[ C - \frac{1}{2} \Delta t \left[ K \right] \right]\right) \{\theta\}_{t-\Delta t}
+ \frac{1}{2} \Delta t \{Q\}_t + \frac{1}{2} \Delta t \{Q\}_{t-\Delta t} .
\]

This equation is written in a more convenient form for the purposes of the solution technique as:

\[
\left[ K \right] \{\theta\}_t = \{Q\}_t ,
\]

where

\[
\{\theta\}_t = \frac{1}{2} \{\theta\}_t + \frac{1}{2} \{\theta\}_{t-\Delta t} ,
\]

\[
\left[ K \right] = \left[ K \right] + \frac{2}{\Delta t} \left[ C \right] ,
\]

\[
\{Q\}_t = \frac{1}{2} \{Q\}_t + \frac{1}{2} \{Q\}_{t-\Delta t} + \frac{2}{\Delta t} \left[ C \right] \{\theta\}_{t-\Delta t} .
\]

7. TWO-DIMENSIONAL TRIANGULAR ELEMENTS

As an example of the application of the method to a specific geometry, the development of the element heat capacity matrix (eq. (14a)), the element conductivity matrix (eq. (14b)) and the element thermal force vector (eq. (14c)) is given for the plane triangular element (fig. 2). The development of these matrices for other types of elements is analogous. The temperature distribution within a typical triangular element \(m\) is assumed to be of the form:

\[
\theta_m(x, y, t) = a_1(t) + xa_2(t) + ya_3(t) .
\]

The constants are expressed in terms of the temperature at the nodal points \(i, j, k\) by:

\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix} = [\beta] 
\begin{pmatrix}
\theta_i \\
\theta_j \\
\theta_k
\end{pmatrix},
\]

where

\[
[\beta] = \begin{pmatrix}
\beta_1 & \cdots & \beta_1 & \cdots & \beta_1 & \cdots \\
\beta_2 & \cdots & \beta_2 & \cdots & \beta_2 & \cdots \\
\beta_3 & \cdots & \beta_3 & \cdots & \beta_3 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

\[
= \frac{1}{2A} \begin{pmatrix}
x_j y_k - x_k y_j \\
y_j y_k - y_k y_j \\
x_k y_i - x_i y_k \\
x_i y_k - x_k y_i \\
\end{pmatrix}
\]

and

\[
A = \frac{1}{2} [x_j(y_k - y_i) + x_i(y_j - y_k) + x_k(y_i - y_j)] .
\]
The temperature gradients, eq. (10), are evaluated for a triangular element by differentiating eq. (29a):

$$\left\{ \begin{array}{c} \theta_i(t) \\ \theta_j(t) \\ \theta_k(t) \end{array} \right\} = \{a_m\} \{b_m(x, y)\}, \quad (30a)$$

where

$$[a_m] = \begin{bmatrix} \cdots \beta_2 i \cdots \beta_2 j \cdots \beta_2 k \cdots \\ \cdots \beta_3 i \cdots \beta_3 j \cdots \beta_3 k \cdots \end{bmatrix}, \quad (30b)$$

The thermal conductivity for two-dimensional flow is:

$$[k^m] = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{xy} & k_{yy} \end{bmatrix}. \quad (31)$$

From eq. (14b) the element conductivity matrix is defined as:

$$[K^m] = \int_{V_m} [a_m]^T [k^m] [a_m] \, dV_m$$

$$= \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & K_{ii}^m & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & K_{jj}^m & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \cdots , \quad (32)$$

where the typical term:

$$K_{rs} = \rho_m t_m (\beta_2 r k_{xx} \beta_2 s + \beta_2 r k_{xy} \beta_3 s + \beta_3 r k_{xy} \beta_3 s) , \quad (33)$$

in which $A_m$ is the area of the element given by eq. (28c) and $t_m$ is the thickness of the element.

The heat capacity matrix for the triangular element is obtained by evaluating eq. (14a):

$$[C^m] = \int_{V_m} \rho_m C_m [b_m]^T [b_m] \, dV_m$$

$$= \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & C_{ii}^m & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & C_{jj}^m & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \cdots , \quad (34)$$

where the typical term is:

$$C_{rs}^m = \int_{V_m} \rho_m C_m b_r b_s \, dV_m . \quad (35)$$

The integration over a triangle of the product of the two linear functions, $b_r$ and $b_s$, can be written as a matrix equation in terms of the value of the functions at the vertices of the triangle.
where the superscripts indicate the nodal point where the function is evaluated. This expression simplifies considerably for the plane element, becoming:

\[
[C]^m = \frac{\rho_m c_m t_m}{12} \begin{bmatrix}
    2 & 1 & 1 \\
    1 & 2 & 1 \\
    1 & 1 & 2
\end{bmatrix},
\]

(36a)

For heat generation within the element the thermal force matrix, eq. (14c), is:

\[
[\mathbf{Q}^m] = \int_{V_m} \rho_m \rho_m \left[ \mathbf{u} \right]^T dV_m = \begin{bmatrix}
    Q_i \\
    \vdots \\
    Q_j \\
    \vdots \\
    Q_k
\end{bmatrix},
\]

(37)

where the typical term \( Q_i^m \) is evaluated as:

\[
Q_i^m = \frac{\rho_m \rho_m t_m A_m}{3} (b_i^1 + b_i^j + b_i^k).
\]

(38)

Equations which include surface heat transfer, eqs. (25a) and (25b), will not be given here since their development is straightforward.

### 8. ADDITIONAL ELEMENTS

Solids of almost any geometric shape may be represented by a combination of different finite elements. The only requirement is that the assumed temperature field is continuous throughout the solid. In addition to the plane triangular element, which is discussed in detail in the previous section, several different elements for the representation of various types of solids are summarized in fig. 3. The use of these elements in heat transfer problems has not been completely investigated; however, most of the proposed elements have been used in the stress analysis of solids [7] and their extension to the heat transfer problem involves the introduction of the scalar temperature field in place of the vector displacement field.

For one-dimensional solids, three are proposed and the temperature field associated with each is as follows:

Two nodal point element:

\[
\theta(x, l) = a_1(l) + x a_2(l).
\]

(39a)

Three nodal point element:

\[
\theta(x, l) = a_1(l) + x a_2(l) + x^2 a_3(l).
\]

(39b)

Four nodal point element:

\[
\theta(x, l) = a_1(l) + x a_2(l) + x^2 a_3(l) + x^3 a_4(l).
\]

(39c)

It is apparent that one-dimensional elements with any number of nodal points are admissible.

In addition to the plane triangular element, several other two-dimensional elements are possible. A compatible temperature field for the rectangular element is:
\( \theta(x, y, t) = \alpha_1(t) + x\alpha_2(t) + y\alpha_3(t) + xy\alpha_4(t) \). (40a)

The practical use of the rectangular element is limited since it cannot be used in approximating curved boundaries. Another possible two-dimensional element is the six-point triangle which is associated with the following temperature field:

\[ \theta(x, y, t) = \alpha_1(t) + x\alpha_2(t) + y\alpha_3(t) + z^2\alpha_4(t) + xy\alpha_5(t) + y^2\alpha_6(t) \]. (40b)

Complete temperature continuity is maintained by the introduction of additional unknown temperatures at the mid-point of each side.

Ring elements with the same cross section as the two-dimensional plane elements may be used in the representation of axisymmetric solids and for the case of axisymmetric temperature distribution, only the evaluation of the volume integrals is different. In the case of non axisymmetric temperature distribution the problem may be expanded in a Fourier series in the circumferential direction \( \phi \). For the three-point triangle the assumed three-dimensional temperature field is of the following form:

\[ \theta(x, y, \phi, t) = \sum_n [\alpha_1^n(t) + x\alpha_2^n(t) + y\alpha_3^n(t)] \sin n\phi \]
\[ + \sum_n [\alpha_1^n(t) + x\alpha_2^n(t) + y\alpha_3^n(t)] \cos n\phi \]. (41)

Admissible temperature fields for three-dimensional elements are as follows:

Rectangular right prism
\[ \theta(x, y, z, t) = \alpha_1(t) + \alpha_2(t)x + \alpha_3(t)y + \alpha_4(t)z + \alpha_5(t)xy + \alpha_6(t)xz + \alpha_7(t)yz + \alpha_8(t)xyz \].

Tetrahedron
\[ \theta(x, y, z, t) = \alpha_1(t) + \alpha_2(t)x + \alpha_3(t)y + \alpha_4(t)z \].

Ten-point tetrahedron
\[ \theta(x, y, z, t) = \alpha_1(t) + \alpha_2(t)x + \alpha_3(t)y + \alpha_4(t)z + \alpha_5(t)x^2 + \alpha_6(t)y^2 + \alpha_7(t)z^2 + \alpha_8(t)xy + \alpha_9(t)xz + \alpha_{10}(t)yz \].

In addition to the elements given in fig. 3, elements may be developed by combining these basic elements. For example, a two-dimensional quadrilateral element may be formed from four triangular elements as shown in fig. 4. The unknown temperature at node 5 of the quadrilateral may be expressed in terms of the temperatures at the other four nodes and eliminated as an unknown in the final set of linear equations.

9. EXAMPLES

To illustrate the solution technique on a transient heat conduction problem for which the exact solution is known, the one-dimensional example of constant heat flux applied to a semi-infinite solid is chosen [8]. If the origin is chosen at the surface of the solid, the material properties chosen to be unity, and the applied heat flux per unit time also chosen to be unity, then the solution is:

\[ \theta(x, t) = \frac{t}{2} \exp \left( -\frac{x^2}{4t} \right) - \frac{1}{2}x \text{erfc} \left( \frac{x}{2\sqrt{t}} \right) \].

The spatial temperature distributions for various times are plotted in fig. 5. Except at early times agreement with the exact solution is excellent. A coarse mesh solution at time \( t = 1.0 \) is given in fig. 6. Also, fig. 6 illustrates the effects of diagonalizing the heat capacity matrix by adding the coefficients of each row and placing the sum on the diagonal. This lumped heat capacity matrix reduces by almost 50\% the number of numerical operations in the step-by-step solution; however, the loss of accuracy appears to be small due to this approximation.

The temperature at \( x = 0 \) versus time is plotted in fig. 7 for two different time increments. The approximate solutions oscillate about the exact solution and converge to the true solution at large times. This type of behavior is typical for very large time increments.

A specific application of interest which incorporates the convection boundary condition and the reduction of the transient solution scheme to a steady-state problem is solved by Peavy [9] using Fourier analysis. An exposed exterior rectangular concrete column (see fig. 8) is subjected to outside weather conditions and an in-
terior environment separated by an abutting wall. The ambient temperature and the surface heat transfer coefficient on the interior face $y=0$ are 100°F and 0.5 Btu/h ft² °F, respectively, and the corresponding quantities on the exterior face $y=l$ are 0°F and 6.0 Btu/h ft² °F. The variation of the ambient temperature and surface heat transfer coefficient on the faces $x=0$ and $x=2a$ is indicated in fig. 8. The thermal conductivity was taken to be 1.0 Btu/h ft °F.

Three problems, representing a column having $l=36$ in., $a=7$ in., and having three different positions for the abutting wall, were solved using a system of 296 nodal points and 252 quad-
Fig. 7. Temperature distribution versus time at $x=0$ indicating stability of solution technique.

Fig. 8. Transverse section of rectangular concrete column with abutting wall separating the inside and outside environment (outside face located at $y=1$).

Fig. 9. Steady state temperatures over the width of the columns against distance measured from the inside surface of the column (positions of abutting walls are shown).
rilateral elements over the half-width of the cross section. The resulting temperature distributions at $x=0$ and $x=a$ as a function of the distance from the interior face are shown in fig. 9. The results of Peary, in terms of a mean temperature over the width of the cross section, are shown on the same figure. The total IBM 7094 time for all three solutions was less than one minute.

10. DISCUSSION

The finite element method has been applied to the transient heat conduction analysis of complex solids. The method possesses unique advantages as compared to other numerical approaches with respect to treating variable distribution of thermal properties, temperature and heat flux boundary conditions and solids of arbitrary geometric shape. The new step-by-step solution technique is stable and provides an efficient digital computer approach for a large class of time-dependent problems. An additional advantage, which has not been demonstrated in this paper, is in the solution of thermal stress problems. The method of temperature analysis presented here is completely compatible with the finite element solution for the stress distribution [7, 10, 11]; hence, both problems may be solved simultaneously within the same computer program.

REFERENCES