Waveguide Solutions by the Finite-element Method

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Summary: A finite-element method based on a function minimization technique is developed for analysing homogeneous waveguide problems. The continuous eigen-value operator as in the Ritz finite-difference method is replaced by a matrix operator. The resulting equations are, however, matrix eigen-value equations. Relevant properties of the matrices are discussed. It is demonstrated that the method has advantages in dealing with awkward boundaries and singularities. A feature of the method is that the error in the eigen-value is a monotonically decreasing function of successive sub-divisions of the cross-section. Minimization of a variational expression assures rapid convergence to the correct eigen-value. Three waveguides are analysed in some detail.

1. Introduction

In recent years numerical techniques have been widely applied to the solution of waveguide problems, where attempts at analytic solutions prove unfruitful. In this paper we formulate a finite-element method for solving propagation problems in uniform homogeneous waveguides. This method has distinct advantages over the standard finite-difference method described by previous authors. In both cases the waveguide cross-section is divided by mesh lines into a number of polygonal sub-domains or elements. In the finite-difference approach, the linear differential equation satisfied by the fields is replaced by a set of difference equations involving field values at mesh points. Elsewhere, the field is undefined. Using a finite-element technique, one first sets up an integral variational expression for some parameter of the problem—in our case the cut-off wave-number. The field is then defined within any element as a linear algebraic function of the fields at vertices of the elements—triangular elements are chosen in our approach. Minimization of the variational expression with respect to the field values at each vertex is carried out, resulting in a set of linear algebraic equations which are written in matrix form as \( A\phi = \lambda B\phi \). The coefficients of \( A \) depend on the area of the elements while those of \( B \) also depend on the area moments about the axis. A finite-element method has been used to obtain solutions of Poisson's equation. The equations described as Ritz finite-difference equations are of the form \( A^T\phi = \theta \) where \( A^T = A \) if the boundary conditions are of the Dirichlet type.

In waveguide problems, the advantages of the finite-element method over the finite-difference method are: (i) successive sub-divisions of the cross-section lead to a monotonic decrease of the eigen-value towards its extremum value, (ii) a more rapid rate of convergence towards the eigen-value is assured, (iii) awkward boundary shapes are easier to handle and do not lead to asymmetric matrices, and (iv) singular points in the waveguide require no special treatment.

The formulation of the finite-element method for waveguides is developed, the properties of the matrices are discussed and three particular problems are solved to illustrate the procedures.

2. Development of the Finite-element Formulation

The basic requirement of the development of finite-element equations for approximating continuous operator eigen-value equations is to find an extremum functional which can be written in Euler density form. As a first step in the development of this method for waveguide problems, a uniform waveguide which is completely filled with homogeneous and isotropic dielectric and whose boundary walls are perfectly conducting is considered.

It is always possible to obtain pure H or E-type modes in homogeneous waveguides. The extremization functional for electromagnetic wave propagation in a homogeneous waveguide is given by

\[
2J(\psi) = \int |\nabla \psi|^2 ds - K^2 \int \psi^2 ds \quad \ldots \quad (1)
\]

where the scalar functions \( \phi \) and \( \psi \) correspond to the longitudinal components of \( \mathbf{H} \) and \( \mathbf{E} \) respectively.

In eqns. (1) and (2) \( K \) is the cut-off wavenumber and is given by

\[
K^2 = \omega^2 \mu \varepsilon - \beta^2 \quad \ldots \quad (3)
\]

It may be shown that the expressions in eqns. (1) and (2) are variational in \( K^2 \).
Instead of approximating the true $\phi$ and $\psi$ by a set of normal mode functions—in many cases we have incomplete knowledge of these functions—the finite-element method employs a set of algebraic functions defined over a sub-section of the whole waveguide cross-section. These sub-sections may be polygonal in shape and are called elements. Thus in the finite-element method the entire domain over which the operator equation is defined is divided into a finite number of elements, on each of which the actual mode function is approximated by a set of continuous algebraic functions which are only defined over the particular element under consideration, and are linearly dependent on the values of $\phi$ or $\psi$ at the vertices of the element. Hence, if an element has $n$ vertices (for a triangular element $n = 3$), the potential $\phi$ or $\psi$ within it may be approximated by

$$\phi(x, y) = \sum_{k} N_k(x, y) \phi_k$$  \hspace{1cm} (4)

where $\phi_k$ and $\psi_k$ are the values of $\phi$ and $\psi$ at the vertex $k$ and $N_k(x, y)$ is a predetermined algebraic function which is uniquely defined and differentiable over the element and which reduces to zero outside the element.

We consider an arbitrary waveguide cross-section with the scheme of grading into elements as shown in Fig. 1.

For illustrative purposes, the elements, numbering $P$, are chosen to be triangular. A typical element (the $e$th) is described by the vertices $i$, $j$, and $m$ in cyclic order. We assume that $\phi_i$, $\phi_j$, and $\phi_m$ are the values of $\phi$ at these vertices. For the element $e$ the functional dependence of $\phi(x, y)$ can be written as

$$\phi^{e}(x, y) = \alpha_0 + \alpha_1 x + \alpha_2 y$$  \hspace{1cm} (5)

where $\alpha_0$, $\alpha_1$, and $\alpha_2$ are to be determined.

If $(x_i, y_i)$, $(x_j, y_j)$, and $(x_m, y_m)$ are the co-ordinates of the vertices $i$, $j$, and $m$ then solving for $\alpha_0$, $\alpha_1$, and $\alpha_2$, we obtain,

$$\phi^{e}(x, y) = \frac{1}{2A_e} [(a_i + b_i x + c_i y)\phi_i + (a_j + b_j x + c_j y)\phi_j + (a_m + b_m x + c_m y)\phi_m]$$  \hspace{1cm} (6)

where

$$a_i = x_j y_m - x_m y_j$$
$$b_i = y_j - y_m$$
$$c_i = x_m - x_j$$  \hspace{1cm} (7)

and $A_e$ is the area of the triangular element which is given by

$$2A_e = \det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix}$$  \hspace{1cm} (8)

The values of other parameters can be obtained by a cyclic rotation of the suffixes $i$, $j$ and $m$.

It is important to note that the functional form of the potential $\phi^{e}(x, y)$ as described in eqn. (6) for all the elements of the entire domain satisfies the continuity relation throughout the whole region. This continuity of $\phi$ is essential for the validity of the variational expression. The derivative normal to the line joining two elements is not, however, continuous for such a functional relation. The finite jump in the normal derivative will not introduce any error into the variational formulation, because the contribution of this type of discontinuity in the normal derivative to the net integrated value of the functional $J$ is always zero.

When the functional form of $\phi$ as given above is substituted into the right-hand side of eqn. (1) and the corresponding integrations are carried out, $J$ will be a function of the variables $\phi_k$. If there are in all $M$ vertices, then

$$J(\phi) = F(\phi_1, \phi_2, \phi_3 \ldots \phi_M)$$  \hspace{1cm} (9)

The optimum value of a set of $\phi_k$ for a certain functional form of $N_k(x, y)$ may be obtained by minimizing the functional given in eqn. (9) with respect to each of $\phi_k$, i.e. equating

$$\frac{\partial J}{\partial \phi_k} = 0; \text{ for } k = 1, 2, 3 \ldots M$$  \hspace{1cm} (10)

In view of the functional form of $\phi^{e}(x, y)$ as given in eqn. (6), eqn. (1) can be rewritten as

$$J(\phi) = \sum_{\epsilon=1}^{p} \int_{\Delta \epsilon} |\nabla \phi|^2 dxdy - K^2 \sum_{\epsilon=1}^{p} \int_{\Delta \epsilon} \phi^2 dxdy$$  \hspace{1cm} (11)

The minimization of $J$ over the entire cross-section is equivalent to the minimization of $J$ over each of the elements individually. Each element contains only three values of $\phi$, but each $\phi_1$ is common to those elements for which the vertex $i$ is common. If we minimize the functional $J$ with respect to each $\phi$ we obtain,
\[ \frac{\partial J}{\partial \psi_i} = \sum_{e=1}^{E} \iint_{\Delta e} \left[ \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial y} \right] \, dx \, dy - K^2 \sum_{e=1}^{E} \iint_{\Delta e} \phi \frac{\partial \phi}{\partial \psi_i} \, dx \, dy \quad \ldots \ldots (12) \]

Substituting for \( \psi (x, y) \) into eqn. (12) and taking into account the fact that the vertex \( i \) is common to the \( R \) elements associated with it, we obtain,

\[ \frac{\partial J}{\psi_i} = \sum_{e=1}^{K} \sum_{k=1}^{R} \phi_k \left[ \frac{\partial N^i_k \partial N^i_k}{\partial x} + \frac{\partial N^i_k \partial N^i_k}{\partial y} \right] \, dx \, dy - K^2 \sum_{e=1}^{R} \sum_{k=1}^{R} N^i_k N^i_k \phi_\Delta \, dx \, dy \quad \ldots \ldots (13) \]

where

\[ N^i_k = (a_k + b_k x + c_k y)/2 \Delta_x \quad \ldots \ldots (14) \]

If we consider the minimization of the functional \( J \) with respect to every \( \psi_i \), we obtain a set of linear algebraic eigen-value equations which in matrix notation can be written as follows:

\[ A \phi = K^2 B \phi \quad \ldots \ldots (15) \]

In eqn. (15) \( A \) and \( B \) are square matrices of order \( M \) and \( \phi \) is a \( (M \times 1) \) column matrix whose components are the values of \( \psi_i \) at the \( M \) vertices. Thus we see that the solution of the above matrix eigen-value problem will give us an approximate solution to all waveguide problems for which eqn. (1) is the appropriate variational functional. Though we have considered above the case of H-mode propagation, the same formulation will hold for E-mode propagation, if the potential function \( \psi \) is replaced by \( \psi \).

3. Evaluation of the Components of the Matrices \( A \) and \( B \)

Consider the partial contribution to the components of the matrix \( A \) by a typical element \( e \). This is obtained from eqn. (13) as follows

\[ \frac{\partial J^e}{\partial \psi_i} \bigg|_A = \iint_{\Delta e} \sum_{k=1}^{R} \phi_k \left[ \frac{\partial N^i_k \partial N^i_k}{\partial x} + \frac{\partial N^i_k \partial N^i_k}{\partial y} \right] \, dx \, dy \quad \ldots \ldots (16) \]

The above equation can be written in compact form by using matrix notation as

\[ \frac{\partial J^e}{\partial \psi_i} \bigg|_A = \iint_{\Delta e} \left( \frac{\partial}{\partial x} (G) \cdot \Phi \right) \frac{\partial N^i_k}{\partial x} + \left( \frac{\partial}{\partial y} (G) \cdot \Phi \right) \frac{\partial N^i_k}{\partial y} \, dx \, dy \quad \ldots \ldots (17) \]

where

\[ \frac{\partial J^e}{\partial \psi_i} \bigg|_A \]

indicates that part of

\[ \frac{\partial J^e}{\partial \psi} \]

which contributes to matrix \( A \) and \( G \) is a row matrix whose components are given by

\[ G = [N^i_k N^i_k N^i_k] \quad \ldots \ldots (18) \]

and \( \Phi^e \) is a column matrix given by

\[ \Phi^e = \begin{bmatrix} \phi^e_i \\ \phi^e_j \\ \phi^e_m \end{bmatrix} \quad \ldots \ldots (19) \]

Substituting the values of \( N^i_k (x, y) \), \( N^i_j (x, y) \) and \( N^i_m (x, y) \) into eqn. (17), we obtain,

\[ \frac{\partial J^e}{\partial \psi_i} \bigg|_A = [S^e_{ii} S^e_{ij} S^e_{im}] \Phi^e \quad \ldots \ldots (20) \]

where

\[ S^e_{ii} = \frac{1}{4 \Delta e} (b_i b_j + c_i c_j) \quad \ldots \ldots (21) \]

In the same way, if \( J \) is minimized with respect to \( \psi_j \) and \( \psi_m \), we obtain,

\[ \begin{bmatrix} \frac{\partial J^e}{\partial \psi_j} \\ \frac{\partial J^e}{\partial \psi_m} \end{bmatrix} \bigg|_A = [S^e_{ji} S^e_{jm} S^e_{mm}] \cdot \Phi^e \quad \ldots \ldots (22) \]

or

\[ \frac{\partial J^e}{\partial \Phi^e} \bigg|_A = \Phi^e \quad \ldots \ldots (23) \]

where

\[ \frac{\partial J^e}{\partial \Phi^e} \bigg|_A \]

is a column matrix given by the left-hand side of eqn. (22).

If the elements are right triangles with vertex \( i \) at the corner of the right angle, then the submatrix \( S^e \) becomes

\[ S^e = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \]

Similarly, we can derive an expression for the partial contribution to the components of the matrix \( B \) subscribed by the element \( e \). This will be

\[ \begin{bmatrix} \frac{\partial J^e}{\partial \psi_i} \\ \frac{\partial J^e}{\partial \psi_j} \\ \frac{\partial J^e}{\partial \psi_m} \end{bmatrix} \bigg|_B = [F^e_{ii} F^e_{ij} F^e_{im}] \Phi^e \quad \ldots \ldots (24) \]

where

\[ F^e_{ij} \]

and

\[ F^e_{im} \]
or
\[
\frac{\partial J^e}{\partial \phi^e} = F^e \cdot \Phi^e \quad \ldots \ldots (25)
\]

where

\[
F_{ij} = \frac{1}{4\Delta e} \int_a^b \int_c^d (a_j + b_j x + c_j y)(a_j + b_j x + c_j y) \, dx dy
\]

and on simplification \( F^e \) becomes

\[
F^e = \frac{\Delta e}{6} \begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 1
\end{bmatrix} \quad \ldots \ldots (26)
\]

Now combining eqns. (23) and (25) we get the partial contribution to the minimization of the function \( J \) subscribed by the approximated function over the element \( e \) as

\[
\frac{\partial J^e}{\partial \Phi^e} = S^e \cdot \Phi^e - K \cdot 2F^e \cdot \Phi^e \quad \ldots \ldots (28)
\]

The matrices \( S^e \) and \( F^e \) as obtained above may be called the element sub-matrices of matrices \( A \) and \( B \) respectively. Though they are written for simplicity as \((3 \times 3)\) square matrices, each of them is a \( (M \times M) \) square matrix with at most nine non-zero components. The matrices \( A \) and \( B \) are the sum of all the element sub-matrices generated by all the elements.

4. Properties of the Matrix Operators \( A \) and \( B \)

The properties of the matrices \( A \) and \( B \) will depend on the nature of \( P \) element sub-matrices as discussed above. Referring back to Section 3 it may be seen that

\[
S_{ij} = S_{ji} \quad \text{and} \quad F_{ij} = F_{ji}
\]

Therefore, each of the element sub-matrices is a symmetric matrix. The matrices \( A \) and \( B \) are the matrix sum of \( P \) symmetric element sub-matrices and thus they are also symmetric. Since each of the vertices is common to a certain number \( R \) of the elements which is much less than the total number \( P \), the generated matrices \( A \) and \( B \) will be sparse band matrices.

It can be shown that the matrix \( B \) as given above is always a positive-definite and diagonally dominant matrix. Equation (27) shows that the components of the element sub-matrix \( F^e \) are all positive and the algebraic sum of the off-diagonal components of any row is equal to the diagonal component. For those vertices which are adjacent to the electric or the magnetic wall in waveguide problems, the boundary conditions on the corresponding potential will generate some rows for which the sum of the off-diagonal terms will be less than the diagonal terms. Therefore, the matrix \( B \) is always diagonally dominant.

Finally, let us consider the quadratic form \( \Phi^T F \Phi \) for an element \( e \):

\[
\Phi^T F \Phi = F_{ij} \phi_i^2 + 2F_{ij} \phi_i \phi_j + F_{jj} \phi_j^2 + 2F_{im} \phi_i \phi_m + 2F_{jm} \phi_j \phi_m + F_{mm} \phi_m^2 \quad \ldots \ldots (29)
\]

Substituting the values of \( F_{ij} \) as given in eqn. (27) into eqn. (29), we obtain

\[
\Phi^T F \Phi = \frac{\Delta e}{12} \left[ (\phi_i + \phi_j + \phi_m)^2 + \phi_i^2 + \phi_j^2 + \phi_m^2 \right] > 0 \quad \ldots \ldots (30)
\]

for all values of non-zero \( \phi \). Therefore the element submatrix \( F^e \) as well as matrix \( B \) is always positive-definite.

Similarly, it can be shown that the diagonal components of all the element submatrices \( S^e \) are positive and the algebraic sum of the off-diagonal terms of any row is exactly equal in magnitude to the diagonal term. Also it can be shown that for triangular elements with interior angles not exceeding 90°, the off-diagonal terms are all negative. Again the boundary conditions on the appropriate potential will make the resultant matrix \( A \) diagonally dominant.

If we now consider the quadratic form \( \Phi^T S \Phi \) for an element submatrix \( S^e \) we obtain,

\[
\Phi^T S \Phi = S_{ii} \phi_i^2 + S_{ij} \phi_j^2 + S_{im} \phi_m^2 + 2S_{ij} \phi_i \phi_j + 2S_{im} \phi_i \phi_m + 2S_{jm} \phi_j \phi_m \quad \ldots \ldots (31)
\]

But,

\[
S_{ii} \geq -(S_{ij} + S_{im}) \quad S_{jj} \geq -(S_{ij} + S_{jm}) \quad S_{mm} \geq -(S_{im} + S_{jm})
\]

Substituting the above inequalities into eqn. (31) we obtain

\[
\Phi^T S \Phi \geq -S_{ij} (\phi_i - \phi_j)^2 - S_{jm} (\phi_j - \phi_m)^2 - S_{im} (\phi_i - \phi_m)^2 \quad \ldots \ldots (32)
\]

Since \( S_{ij} S_{im} \) and \( S_{jm} \) are negative,

\[
\Phi^T S \Phi > 0 \quad \text{for all non-zero column vector} \ \Phi.
\]

Hence the element submatrices \( S^e \) and matrix \( A \) are also positive-definite. These properties of matrices \( A \) and \( B \) guarantee the convergence of iterative solutions.

5. Particular Solutions

The numerical techniques developed in the previous sections are used to calculate the wave number of three typical waveguide problems. These problems are chosen to demonstrate the form of the finite-element equations in special cases, the method of handling awkward boundaries and singularities in the waveguide cross-section.

5.1. Circular Waveguide

If in a two-dimensional boundary value problem, the functional variation of the fields with respect to any
one of the dimensions is known, then the corresponding extremum functional can be written in terms of only one variable. For a circular waveguide the extremum functional is given by

\[
J(\phi) = \int_0^\pi \left( \frac{\partial \phi}{\partial r} \right)^2 r \, \mathrm{d} \theta \, \mathrm{d}r + \int_0^\pi \frac{1}{r} \left( \frac{\partial \phi}{\partial \theta} \right)^2 \, \mathrm{d} \theta \, \mathrm{d}r - K^2 \int_0^r \phi^2 r \, \mathrm{d}r \, \mathrm{d}r
\]

where \( \phi \) is the longitudinal component of the appropriate field vector. In a circular waveguide, the azimuthal variation of the fields takes the form \( \cos n\theta \sin m\theta \). If we insert the above form of \( \theta \)-dependence for \( \phi \) into eqn. (32) and integrate with respect to \( \theta \), then the resultant integrals will be functions of \( r \) only. We now divide the radius into a number of line segments as shown in Fig. 2, each of these segments being the element in a one-dimensional problem. We assume a linear variation of the field over each element and following the procedure discussed earlier, we obtain a set of linear algebraic equations. The solution for the lowest eigen-value of this set of equations will correspond to the wave numbers for \( \mathbf{H}_{n1} \) or \( \mathbf{E}_{n1} \) mode. To illustrate this, a few of the typical finite-element equations for \( \mathbf{E}_{01}, \mathbf{E}_{11} \) and \( \mathbf{H}_{11} \) modes, corresponding to the scheme of grading as shown in Fig. 2, are given below. Matrices \( A, B \) and \( \phi \) are given for the case \( N = 4 \).

(a) \( \mathbf{E}_{01} \) mode

If \( p \) is an interior vertex then

\[
-(2p-1)\phi_{p-1} + 4p\phi_p - (2p+1)\phi_{p+1} = \frac{K^2 h^2}{6} \times
\]

\[
[2(p-1)\phi_{p-1} + 8p\phi_p + (2p+1)\phi_{p+1}]
\]

and for the vertex at the centre of the guide

\[
\phi_0 - \phi_1 = \frac{K^2 h^2}{6} (\phi_0 + \phi_1)
\]

where \( h \) is the length of each element.

\[
A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 12 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 8 & 3 & 0 \\ 0 & 3 & 16 & 5 \\ 0 & 0 & 5 & 24 \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}
\]

(b) \( \mathbf{E}_{11} \) mode

For \( \mathbf{E}_{11} \) mode all vertices are interior vertices and for any vertex \( p \) we obtain

\[
A = \begin{bmatrix} 4 & 2 & -2 & 0 \\ -2 & 2 & 4 + 2 & 0 \\ 0 & -6 & 2 & 4 + 16 & 2 \\ 0 & 0 & -16 & 1 & 24 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 3 & 0 \\ 3 & 16 & 5 \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}
\]

(c) \( \mathbf{H}_{11} \) mode

For all interior vertices the finite-element equations will be the same as eqn. (35), but for the vertex \( N \), at the guide wall, this becomes

\[
- \left[ N(N-1) \ln N \right] \frac{N}{N-1} \phi_{N-1} +
\]

\[
+ \left[ \frac{1}{2} (N-1)^2 \ln N \right] \phi_N = \frac{K^2 h^2}{12} \times
\]

\[
[2(N-1) \cdot \phi_{N-1} + (4N-1)\phi_N]
\]

\[
A = \begin{bmatrix} 4 & 2 & -2 & 0 \\ -2 & 2 & 4 + 2 & 0 \\ 0 & -6 & 2 & 4 + 16 & 2 \\ 0 & 0 & -16 & 1 & 24 \end{bmatrix}, \quad B = \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}
\]
Numerical computations of $K_a$ for a circular waveguide sustaining $E_{01}$, $E_{11}$ and $H_{11}$ modes are given in Tables 1, 2 and 3.

5.2. Boundary Matching

The theory developed in this paper can be suitably applied to problems with arbitrary boundaries. In the finite-element method, an arbitrary boundary is approximated by a set of piece-wise linear boundaries. The common points between successive linear boundaries are considered as the boundary vertices. The resultant eigen-value operators remain symmetric. As an illustration, a circular waveguide is analysed numerically in cartesian co-ordinates for the $E_{01}$ mode, and the curved boundary is divided up as shown in Figs. 3(a), (b) and (c). Numerical results are shown in Table 4. Comparing the computed results with the exact result we find that the rate of convergence is fairly rapid.

5.3. Dominant Mode in a Single Ridged Waveguide

Exact analytical solutions in compact form cannot be obtained for ridged waveguides. In writing the finite-difference equations, various types of assumptions are necessary in taking into account the presence of the re-entrant corner. No special treatment is necessary for such a point in the finite-element method. The finite-element equation for a typical vertex 0 and for the vertex at the singular point $s$ of a ridged waveguide graded into triangular elements as shown in Fig. 4 are given below.

**Table 1:** $E_{01}$ mode in a circular waveguide

<table>
<thead>
<tr>
<th>$h/a$</th>
<th>$K_a$</th>
<th>$h/a$</th>
<th>$K_a$</th>
<th>$h/a$</th>
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<td>$1/2$</td>
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<td>$1/2$</td>
<td>2.408</td>
<td>$1/2$</td>
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</tr>
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<td>$1/2$</td>
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<td>$1/2$</td>
<td>2.408</td>
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<td>2.405</td>
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</table>

First root of $J_0(K_a)$ is $K_a = 2.405$

**Table 2:** $E_{11}$ mode in a circular waveguide

<table>
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<th>$K_a$</th>
<th>$h/a$</th>
<th>$K_a$</th>
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<td>$1/3$</td>
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<tr>
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<td>3.841</td>
</tr>
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</table>

First root of $J_1(K_a)$ is $K_a = 3.832$

**Table 3:** $H_{11}$ mode in a circular waveguide

<table>
<thead>
<tr>
<th>$h/a$</th>
<th>$K_a$</th>
<th>$h/a$</th>
<th>$K_a$</th>
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<td>1.843</td>
</tr>
<tr>
<td>$1/2$</td>
<td>1.856</td>
<td>$1/2$</td>
<td>1.845</td>
<td>$1/2$</td>
<td>1.842</td>
</tr>
</tbody>
</table>

First root of $J_1(K_a)$ is $K_a = 1.841$

**Table 4:** $E_{01}$ mode in a circular waveguide

<table>
<thead>
<tr>
<th>$h/a$</th>
<th>Fig. No.</th>
<th>Number of equations</th>
<th>$K_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/4$</td>
<td>3(a)</td>
<td>3</td>
<td>2.631</td>
</tr>
<tr>
<td>$1/3$</td>
<td>3(b)</td>
<td>9</td>
<td>2.511</td>
</tr>
<tr>
<td>$1/3$</td>
<td>3(c)</td>
<td>31</td>
<td>2.460</td>
</tr>
</tbody>
</table>

Extrapolated $K_a = 2.420$. First root of $J_0(K_a)$ is $K_a = 2.405$
For a typical vertex 0

\[ 4\phi_0 - \phi_1 - \phi_2 - \phi_3 - \phi_4 = \frac{K^2 h^2}{12} (6\phi_0 + \phi_4 + \phi_2 + \phi_3 + \phi_4 + \phi_4) \]

and for the vertex at the singular point \( s \)

\[ 3\phi_s - \phi_0 - \phi_0 - \frac{1}{2}\phi_s - \frac{1}{2}\phi_s = \frac{K^2 h^2}{12} (4\phi_0 + \phi_0 + \phi_0 + \phi_0 + \frac{1}{2}\phi_s + \frac{1}{2}\phi_s) \]

Table 5 gives the numerical computation of the dominant mode wave number for such a guide. A study of these results shows that on successive element sub-division the wave number monotonically approaches the value given by Pyle.  

Table 5: Dominant mode in a single ridged rectangular waveguide

<table>
<thead>
<tr>
<th>( h / a )</th>
<th>18</th>
<th>60</th>
<th>216</th>
<th>816</th>
<th>3168</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of vertices</td>
<td>2-300</td>
<td>2-274</td>
<td>2-257</td>
<td>2-252</td>
<td>2-250</td>
</tr>
<tr>
<td>( Ka )</td>
<td>6.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. Conclusions

By means of the finite-element approach, a uniform treatment of the boundary conditions and the differential equation is possible without any special consideration of the detailed nature of the boundary. The coefficients of the matrices \( A \) and \( B \) are developed from the moments of the element area about the axes and are thus insensitive to the variations of the local slope of the boundary. Under all circumstances, the matrices remain symmetric, positive-definite, sparse and are band-type; these properties are of great practical value when solutions for very large matrices are required via the computer and also ensure convergence of an iterative solution using over-relaxation procedures.

The rate of convergence is dependent on the over-relaxation factor used during the computation. This factor is calculated via the Rayleigh coefficient after every few iterations and thus an optimum rate of convergence is assured. Since the computed eigen-value is an upper bound to the exact value and is also derived from the variational expression, successive mesh sub-division leads rapidly to the correct eigen-value. If successive approximations to the exact eigen-value are known, eigen-value extrapolation may also be used successfully.

The method can be extended to deal with inhomogeneous waveguide problems.

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References