Modeling Convective Heat Transfer under Laminar Flow of a Newtonian Fluid in Simple Geometries

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Abstract: This paper reports on a series of studies using COMSOL Multiphysics, carried out to investigate the phenomena of fully developed and developing laminar flows of a Newtonian fluid initially at constant temperature, between parallel plates and through circular tubes where heat exchange takes place at the interface between the fluid and the plates or the tube wall. Selected numerical results were compared with existing analytical solutions that were in turn computed using Maple. The validated COMSOL models constitute excellent teaching and learning tools that contribute to and facilitate understanding of complex convection heat transfer problems.

Keywords: Laminar Flow, Newtonian Fluid, Convective Heat Transfer.

1. Introduction

The study of combined fluid flow and heat transfer between flat plates and inside tubes is a fundamental problem in convective heat transfer (1,2). Much insight about flow phenomena and energy transport processes can be gained from an understanding of the behavior of these simple systems. Since analytical solutions are available for a few selected problems, a suitable baseline for the validation of numerical computations can be readily obtained.

2. Governing Equations

2.1 Laminar Flow of a Newtonian Fluid

The governing equations for the flow of a Newtonian fluid are the mass balance equation (continuity equation)

\[ \text{div } \mathbf{v} = 0 \]

where \( \mathbf{v} = (u, v, w) \) is the velocity vector, and the momentum balance equations (equations of motion)

\[ \rho \ v \cdot \text{grad} \ u = - \frac{\delta P}{\delta x} + \mu \ \text{div grad } u \]

\[ \rho \ v \cdot \text{grad} \ v = - \frac{\delta P}{\delta y} + \mu \ \text{div grad } v \]

\[ \rho \ v \cdot \text{grad} \ w = - \frac{\delta P}{\delta z} + \mu \ \text{div grad } w \]

where \( \rho \) is the fluid density, \( P \) the pressure and \( \mu \) the fluid viscosity.

Consider now a Newtonian fluid (e.g. water) flowing between two parallel plates. Adopt a rectangular Cartesian system of coordinates x-y and let the plates be horizontal and of width \( L \) along the x-direction. Let the separation between the plates in the vertical direction be \( D=2a \). A simple case is that of fully developed flow. Here, solving the momentum balance equation (with \( y=0 \) at the gap centerline) yields the x-velocity profile in the gap as

\[ u = \frac{1}{2} \mu \ \frac{\delta P}{\delta x} \left( a^2 - y^2 \right) \]

For the case of fully developed flow inside a tube of diameter \( D=2R \), we adopt a cylindrical polar system of coordinates with the x direction along the cylinder axis and r along the radial direction. The corresponding solution of the momentum balance equation (with \( r=0 \) at the tube centerline) gives the axial (x-directed) velocity profile inside the tube as

\[ u = \frac{1}{4} \mu \ \frac{\delta P}{\delta x} \left( R^2 - r^2 \right) \]

A more general case is that when the fluid enters the gap between the plates or the tube, from a larger reservoir and with a given velocity profile at the inlet. In this case the flow field develops from the given profile at the inlet into the fully developed profile given by the above
equation and the problem is more difficult to solve. A key question in this case is about the hydrodynamic entrance length of the flow $L_e$, i.e. the distance from the inlet until the flow becomes fully developed. This is also the distance measured from the inlet at which the hydrodynamic boundary layers that form on the conduit walls, meet at the centerline of the gap. Specifically, for the case of developing flow inside a tube, the entrance length has been given as

$$L_{e,t} = 0.1 \frac{N_{Re} R}{N_{Pr}}$$

Where $N_{Re}$ is the Reynolds number defined as

$$N_{Re} = \frac{\rho D V}{\mu}$$

where $V$ is the average velocity of the fluid in the gap (3).

2.2 Convective Heat Transfer

If the fluid entering the gap between the plates or the tube is at a different temperature from that of the conduit walls, an exchange of thermal energy takes place. Energy transfer takes place in the fluid by molecular transport (conduction) and by the macroscopic motion of the fluid itself (convection). The mathematical statement of the problem involves the differential thermal energy balance equation which, under steady state conditions is

$$v \cdot \nabla T = \alpha \nabla^2 T$$

Where $\alpha = k/\rho C_p$ is the thermal diffusivity ($k$ = thermal conductivity; $C_p$ = specific heat) and $T$ is the fluid temperature.

For the case of flow between parallel plates, the energy equation reduces to

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

where $u$ and $v$, are, respectively the x- and y-components of velocity.

And for the case of the flow inside a tube, it becomes

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial r} = \alpha \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial x^2} \right)$$

where $u$ and $v$, are here the x- (axial) and r- (radial) components of velocity, respectively.

In both instances above, the x-direction is the main direction of the flow (parallel to the conduit walls). Simpler forms are readily obtained for the important case when the second term on the left hand sides of both equations above can be neglected for being small compared to the first one (velocity parallel to the conduit walls is dominant).

A corresponding thermal entrance length can be defined as the distance along the flow direction, measured from the inlet, at which the thermal boundary layers that form on the conduit walls as a result of the heat exchange, meet at the centerline of the gap. Again, specifically, for the case of developing flow inside a pipe, the thermal entrance length has been given as

$$L_{e,t} = 0.1 \frac{N_{Re} N_{Pr} R}{N_{Pr}}$$

Where $N_{Pr}$ is the Prandtl number defined as

$$N_{Pr} = \frac{\nu}{\alpha}$$

with $\nu = \mu/\rho$ being the kinematic viscosity (5).

As an example of an existing analytical solution, consider the fully developed laminar flow of a uniform temperature fluid ($T=T_o$) inside a tube. At some point along the flow, the temperature at the inner surface of the tube suddenly becomes $T_s$, different from the fluid temperature and is maintained there subsequently. This is sometimes called the Graetz problem (4) and an analytical solution in the form of infinite series has been obtained and for $r=0$ (i.e. the pipe centerline) is given by

$$T(r=0,x) = T_o + (T_o - T_s) F(\alpha/R, U)$$

Where the first three terms of the function $F$ are given by

$$F = 1.477 \exp(-3.6585 \alpha/R^2 U) - 0.81 \exp(-22.1778 \alpha/R^2 U) + 0.385 \exp(-53.045 \alpha/R^2 U)$$

To compute the fluid temperature at radial locations other than $r=0$, each term in the above expression for $F$ must be multiplied by functions $R_i$ (i=0,1,2,...) that can also be determined theoretically. We have used the symbolic manipulation program Maple to calculate numerical values of the temperature inside the tube from the general series solution and used
the results as a baseline for comparison against numerical computation using the finite element method in COMSOL.

3. Numerical Models

3.1 Convective Heat Transfer for Flow between Parallel Plates

Using COMSOL Multiphysics a 2D domain was created (length $L = 0.01$ m, height $a = 0.001$ m) to represent the fluid contained between a plate wall and the centerline of the gap between the plates. No-slip (zero velocity) was imposed at the plate wall ($y=0$), with symmetry assumed at the gap centerline ($y=a$). At the inlet ($x=0$) either a given uniform velocity was assumed (say $u_o = 0.025$ m/s) or the corresponding fully developed profile was used. Finally at the exit, an outlet (zero pressure) condition was imposed. For the thermal problem, the fluid temperature at the inlet ($x=0$) was assumed uniform ($T_o = 298$ K), whereas the wall temperature was assumed as constant ($T_s = 325$ K). Further, thermal symmetry was assumed at the gap centerline ($y=a$) and a convective flux condition was used at the exit. The fluid was assumed to be water and the property values used were: $\rho = 10^3$ kg/m$^3$; $\mu = 10^{-3}$ Pa-s; $k = 0.597$ W/m K; and $C_p = 420$ J/kg K. Although the above parameter values were used for illustration in the computations below, the COMSOL model is generic and can readily be modified to investigate many other scenarios. This powerful feature of COMSOL facilitates significantly continued exploration and learning about convective heat transfer problems. Whenever possible, the computed results were compared against existing analytical solutions. In all other cases, mesh independence of the computed results was checked by an h-extension procedure consisting of systematic mesh refinement and repetition of the calculations (6).

Additional information and descriptions of several other problems that were investigated during this study can be found in Ms. Baron’s Master’s Project report (7).

3.2 Convective Heat Transfer for Flow inside a Tube

Using COMSOL Multiphysics a 2D axisymmetric domain was created (pipe length $L = 0.01$ m, pipe radius $R = 0.001$ m) to represent the fluid contained between the pipe wall and the centerline of the tube. No-slip (zero velocity) was imposed at the plate wall ($r=R$) and symmetry was assumed at the pipe centerline ($r=0$). At the pipe inlet ($x=0$) either a given uniform velocity was assumed (say $u_o = 0.01$ m/s) or the corresponding fully developed profile was used. Finally at the exit, an outlet (zero pressure) condition was imposed. For the thermal problem, the fluid temperature at the inlet ($x=0$) was assumed uniform ($T_o = 298$ K), whereas the wall temperature was assumed as constant ($T_s = 325$ K). Further, thermal symmetry was assumed at the pipe centerline ($r=0$) and a convective flux condition was used at the exit. Again, the working fluid was assumed to be water and the same values of physical properties were used.

4. Results and Discussion

Figure 1 shows computed results of velocity and temperature for the developing flow between parallel plates using 4480 elements. Although the computed results become practically mesh independent for a much smaller number of elements, this calculation took only about 11 seconds on a rather ordinary PC. The specific values of the Reynolds and Prandtl numbers used were $N_{Re} = 50$ and $N_{Pr} = 0.7$. Since $N_{Pr} < 1$ the thermal boundary layer is expected to develop faster than the hydrodynamic boundary layer and this is indeed obtained (i.e $L_{e,t} \approx 0.005$ vs. $L_{e,h} \approx 0.001$). Moreover, as indicated above, both entrance layers are expected to increase linearly with the value of $N_{Re}$; this is indeed obtained with the COMSOL model.

![Figure 1. Calculated velocity profile and temperature field for developing flow between parallel plates.](image)
Figure 2 shows computed results of velocity and temperature for the developing flow inside a tube using 4736 elements. As in the parallel plate case, the hydrodynamic entrance length here is longer than the thermal entrance length and both entry lengths increase in linear proportion with the value of $N_{Re}$.

Figure 2. Calculated velocity profile and temperature field for developing flow inside a tube.

5. Conclusions

COMSOL Multiphysics provides a robust and easy to use computer-based environment to stimulate learning and understanding of convective heat transfer processes. Excellent agreement with established know-how about laminar convective heat transfer problems was readily obtained. Although we restricted our studies to laminar flow problems, we have demonstrated that the software can be used with confidence to support not only foundational work at undergraduate and graduate levels, but also in research projects dealing with more specialized aspects of the problem. Additional detail about this work, including the COMSOL programs and reports can be found on the associated web page (8).

6. References