VIBRATION PROBLEMS IN ENGINEERING
Fourth Edition

S. TIMOSHENKO
Professor Emeritus of Engineering Mechanics
Stanford University

D. H. YOUNG
Professor Emeritus of Civil Engineering
Stanford University

W. WEAVER, JR.
Professor of Structural Engineering
Stanford University

JOHN WILEY & SONS
NEW YORK LONDON SYDNEY TORONTO
In cases where the boundary is different from those discussed above, the investigation of vibrations presents greater mathematical difficulties. However, the case of an elliptical boundary has been completely solved.†

5.24 Transverse Vibrations of Plates

Figure 5.39a shows a plate with a uniform thickness $h$ that is assumed to be small in comparison with its other dimensions. We take the $x$-$y$ plane as the middle plane of the plate and assume that deflections in the $z$ direction are small in comparison with the thickness $h$. In addition, the normals to the middle plane of the plate are assumed to remain normal to the deflected middle surface during vibrations.

Let us consider the strains in a thin layer of a typical element, as indicated by the shaded area at the distance $z$ from the middle surface in Fig. 5.39b. These strains will be represented by the following equations:‡

$$
\epsilon_x = -z \frac{\partial^2 v}{\partial x^2} \quad \epsilon_y = -z \frac{\partial^2 v}{\partial y^2} \quad \gamma_{xy} = -2z \frac{\partial^2 v}{\partial x \partial y} \quad (a)
$$

In these expressions $v$ denotes the deflection of the plate in the $z$ direction; and $\epsilon_x$, $\epsilon_y$, and $\gamma_{xy}$ are the normal strains and the shear strain in the thin layer. The stresses corresponding to these strains are determined by the known relationships†

$$
\sigma_x = \frac{Ez}{1 - \nu^2} (\epsilon_x + \nu \epsilon_y) = \frac{-Ez}{1 - \nu^2} \left[ \frac{\partial^2 v}{\partial x^2} + \nu \frac{\partial^2 v}{\partial y^2} \right] \\
\sigma_y = \frac{Ez}{1 - \nu^2} (\epsilon_y + \nu \epsilon_x) = \frac{-Ez}{1 - \nu^2} \left[ \frac{\partial^2 v}{\partial y^2} + \nu \frac{\partial^2 v}{\partial x^2} \right] \\
\tau_{xy} = G\gamma_{xy} = -\frac{Ez}{(1 + \nu)} \frac{\partial^2 v}{\partial x \partial y}
$$

in which $\nu$ denotes Poisson's ratio.

The potential energy accumulated in the shaded layer of the element during deformation will be

$$
dU = \frac{(\epsilon_x^2 + \epsilon_y^2 + \gamma_{xy}^2)}{2} \, dx \, dy \, dz 
$$

Substitution of eqs. (a) and (b) into this expression yields

$$
dU = \frac{Ez^2}{2(1 - \nu^2)} \left[ \left( \frac{\partial^2 v}{\partial x^2} \right)^2 + \left( \frac{\partial^2 v}{\partial y^2} \right)^2 \right] + 2\nu \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 v}{\partial x \partial y} + 2(1 - \nu) \left( \frac{\partial^2 v}{\partial x \partial y} \right)^2 \, dx \, dy \, dz \quad (c)
$$

Integrating eq. (c) over the volume of the plate, we obtain the potential energy of bending as

$$
U = \int \int \int dU = \frac{D}{2} \int \int \left[ \left( \frac{\partial^2 v}{\partial x^2} \right)^2 + \left( \frac{\partial^2 v}{\partial y^2} \right)^2 \right] + 2\nu \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 v}{\partial x \partial y} + 2(1 - \nu) \left( \frac{\partial^2 v}{\partial x \partial y} \right)^2 \, dx \, dy \quad (5.174)
$$

where $D = Ekh^3/[12(1 - \nu^2)]$ is the flexural rigidity of the plate.

The kinetic energy of a transversely vibrating plate will be

$$
T = \frac{\rho h}{2} \int \int \oint \dot{v}^2 \, dx \, dy \quad (5.175)
$$

where $\rho h$ is the mass per unit area. These expressions for $U$ and $T$ will now be applied to particular types of plates.

Rectangular Plates. In the case of a rectangular plate (see Fig. 5.39a) with simply supported edges, we can proceed as in the case of a rectangular


‡ It is assumed that there is no stretching of the middle plane.

† E. Mathieu, J. Math. (Liouville), Vol. 13, 1868.
membrane and take the deflection of the plate during vibration as the double series

\[ v = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]  

which are the normal functions for the case under consideration. It is easy to see that each term of this series satisfies the conditions at the edges, which require that \( v = \partial^2 v / \partial x^2 = 0 \) at \( x = 0 \) and \( x = a \); and \( v = \partial^2 v / \partial y^2 = 0 \) at \( y = 0 \) and \( y = b \). If eq. \((d)\) is substituted into eq. \((5.174)\), the following expression for the potential energy will be obtained:

\[ U = \frac{\pi^4 ab}{8} D \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn}^2 \left( \frac{m^4}{a^4} + \frac{n^4}{b^4} \right) \]  

(5.176)

In addition, the kinetic energy (eq. \(5.175\)) becomes

\[ T = \frac{\rho ab}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn}^2 \]  

(5.177)

The inertial force for a typical element of the plate is \(-p\delta \, dx \, dy\). Proceeding as before and considering a virtual displacement

\[ \delta v_{mn} = \delta \phi_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]

we obtain the differential equation of motion for free vibrations in principal coordinates to be

\[ \rho h \phi_{mn} + \pi^4 D \phi_{mn} \left( \frac{m^4}{a^4} + \frac{n^4}{b^4} \right) = 0 \]

The solution of this equation is

\[ \phi_{mn} = C_1 \cos pt + C_2 \sin pt \]

where

\[ p = \pi^2 \sqrt{\frac{D}{\rho h}} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \]  

(5.178)

From this formula the frequencies of vibration can be easily calculated. For example, in the case of a square plate, we obtain for the lowest mode of vibration

\[ f_1 = \frac{p_1}{2\pi} = \frac{\pi}{a} \sqrt{\frac{D}{\rho h}} \]  

(5.179)

In considering higher modes of vibration and their nodal lines, we see that the discussion previously given for the vibrations of a square membrane (see Fig. 5.36) apply equally well to a square plate. In addition, the case of forced vibrations of a rectangular plate with simply supported edges can be solved without difficulty. It should be noted that the vibrations of a rectangular plate, for which two opposite edges are supported while the other two edges are free or clamped, can also be analyzed without great mathematical difficulty.†

However, vibrational analyses of plates with all edges free or clamped are much more complicated. For the solution of these problems, Ritz's method has been found to be very useful.‡ To apply this method, we assume

\[ v = Z \cos(pt - \alpha) \]  

(5.174)

where \( Z \) is a function of \( x \) and \( y \) which approximates the mode of vibration. Substituting eq. \((e)\) into eqs. \((5.174)\) and \((5.175)\), we obtain the following expressions for the maximum potential and kinetic energies of vibration:

\[ U_{\text{max}} = \frac{D}{2} \int \left[ \left( \frac{\partial^2 Z}{\partial x^2} \right)^2 + \left( \frac{\partial^2 Z}{\partial y^2} \right)^2 + 2 \nu \frac{\partial^2 Z}{\partial x^2} \frac{\partial^2 Z}{\partial y^2} \right] \, dx \, dy \]

\[ T_{\text{max}} = \frac{\rho h}{2} \int Z^2 \, dx \, dy \]

Equating these expressions and solving for \( p^2 \), we find

\[ p^2 = \frac{2}{\rho h} \int \frac{U_{\text{max}}}{Z^2} \, dx \, dy \]  

(5.180)

Now we take the function \( Z \) in the form of a series

\[ Z = a_1 \Phi_1(x, y) + a_2 \Phi_2(x, y) + a_3 \Phi_3(x, y) + \ldots \]  

(5.181)

each term of which satisfies the conditions at the boundary of the plate. It is then necessary to determine the coefficients \( a_1, a_2, a_3, \ldots \) in such a manner as to make the result of eq. \((5.180)\) a minimum. In this way we arrive at a system of equations of the type

\[ \frac{\partial}{\partial a_n} \int \left[ \left( \frac{\partial^2 Z}{\partial x^2} \right)^2 + \left( \frac{\partial^2 Z}{\partial y^2} \right)^2 + 2 \nu \frac{\partial^2 Z}{\partial x^2} \frac{\partial^2 Z}{\partial y^2} \right] \, dx \, dy = 0 \]