Appendix C

Properties of the stress tensor

Some of the basic properties of the stress tensor and traction vector are reviewed in the following.

C.1 The traction vector

Let us assume that the state of stress at a point is known and let us determine the components of the traction vector $T_x$, $T_y$, $T_z$ acting on the inclined face of the infinitesimal tetrahedral volume element, as shown in Fig. C.1. By definition, the traction vector, also known as the stress vector, is

$$ \vec{F} = \lim_{\Delta A \to 0} \frac{\Delta \vec{F}}{\Delta A} $$

where $\Delta \vec{F}$ is the differential force, acting on the inclined face of the tetrahedron, the area of which is $\Delta A$. This force is in equilibrium with the resultants of the stresses acting on the other three faces of the tetrahedron.

We first show that the unit normal to the inclined plane, denoted by $\vec{n}$, has the components

$$ n_x = \frac{\Delta A_x}{\Delta A}, \quad n_y = \frac{\Delta A_y}{\Delta A}, \quad n_z = \frac{\Delta A_z}{\Delta A} \quad (C.1) $$

where $\Delta A_x$ (resp. $\Delta A_y$, $\Delta A_z$) is the area of that face of the tetrahedron to which the $x$ axis (resp. $y$, $z$ axis) is normal, see Fig. C.1(a). Consider the cross product:

$$ \vec{c} = \vec{a} \times \vec{b} = (\Delta y \vec{e}_y - \Delta x \vec{e}_x) \times (\Delta z \vec{e}_z - \Delta y \vec{e}_y) $$

$$ = \Delta y \Delta z \vec{e}_x + \Delta x \Delta z \vec{e}_y + \Delta x \Delta y \vec{e}_z = 2 \Delta A_x \vec{e}_x + 2 \Delta A_y \vec{e}_y + 2 \Delta A_z \vec{e}_z $$

where $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ are the unit basis vectors. The vector $\vec{c}$ is normal to the inclined plane and its absolute value is $2\Delta A$. Therefore $\vec{n} = \vec{c}/|\vec{c}| = \vec{c}/2\Delta A$ and the components of the unit normal are as given in eq. (C.1).
The equations of equilibrium are:
\[
\begin{align*}
\sum \Delta F_x &= 0 : & T_x \Delta A - \sigma_x \Delta A_x - \tau_{yx} \Delta A_y - \tau_{xz} \Delta A_z &= 0 \\
\sum \Delta F_y &= 0 : & T_y \Delta A - \tau_{xy} \Delta A_x - \sigma_y \Delta A_y - \tau_{yz} \Delta A_z &= 0 \\
\sum \Delta F_z &= 0 : & T_z \Delta A - \tau_{xz} \Delta A_x - \tau_{yz} \Delta A_y - \sigma_z \Delta A_z &= 0.
\end{align*}
\]

On dividing by \(\Delta A\), and making use of the fact that the stress tensor is symmetric, we have:
\[
\begin{pmatrix}
T_x \\
T_y \\
T_z
\end{pmatrix} = \begin{bmatrix}
\sigma_x & \tau_{yx} & \tau_{xz} \\
\tau_{xy} & \sigma_y & \tau_{yz} \\
\tau_{xz} & \tau_{yz} & \sigma_z
\end{bmatrix} \begin{pmatrix}
n_x \\
n_y \\
n_z
\end{pmatrix}.
\] (C.2)

In index notation this can be written as:
\[
T_i = \sigma_{ij} n_j. 
\] (C.3)

### C.2 Principal stresses

Let us now ask whether it would be possible to find a plane such that the traction vector, acting on that plane, would be normal to the plane (i.e., the shearing components would be zero). This condition is fulfilled when
\[
\sigma_{ij} n_j = T n_i \equiv T \delta_{ij} n_j
\]
where \(T\) is the magnitude of the traction vector. This is a characteristic value problem:
\[
(\sigma_{ij} - T \delta_{ij}) n_j = 0.
\] (C.4)
Since $\sigma_{ij}$ is symmetric, all characteristic values are real. The characteristic values are called principal stresses and the normalized characteristic vectors are the unit vectors that define the directions of principal stresses. Since the characteristic vectors are mutually orthogonal, in every point there is an orthogonal coordinate system in which the stress state is characterized by normal stresses only. This coordinate system is uniquely defined only when all characteristic values are simple.

It follows from eq. (C.4) that the principal stresses are the roots of the following characteristic equation:

$$T^3 - I_1 T^2 - I_2 T - I_3 = 0$$  \hfill (C.5)

where

$$I_1 = \sigma_{kk}, \quad I_2 = \frac{1}{2}(\sigma_{ij} \sigma_{ij} - \sigma_{ii} \sigma_{jj}), \quad I_3 = \det(\sigma_{ij}).$$  \hfill (C.6)

The principal stresses characterize the state of stress in a point and hence do not depend on the coordinate system in which the stress components are given. Therefore the coefficients $I_1$, $I_2$ and $I_3$ are invariant with respect to rotation of the coordinate system. These coefficients are called the first, second and third stress invariant respectively.

### C.3 Transformation of vectors

Consider a Cartesian coordinate system $x'_i$ rotated relative to the $x_i$ system as shown in Fig. C.2. Let $\alpha_{ij}$ be the angle between the axis $x'_i$ and the axis $x_j$ and let

$$g_{ij} := \cos \alpha_{ij}. \hfill (C.7)$$

In other words, the $i$th row of $g_{ij}$ is the unit vector in the direction of axis $x'_i$ in the unprimed system. Therefore if $a_i$ is an arbitrary vector in the unprimed system then the same vector in the primed system is:

$$a'_i = g_{ij} a_j. \hfill (C.8)$$

Conversely, the $j$th column of $g_{ij}$ is the unit vector in the direction of axis $x_j$ in the primed system. Therefore if $a'_r$ is an arbitrary vector in the primed system then the same vector in the unprimed system is:

$$a_r = g_{kr} a'_k. \hfill (C.9)$$

Given the definition of $g_{ij}$ and the orthogonality of the coordinate systems, $g_{ij}$ multiplied by its transpose must be the identity matrix:

$$g_{ri} g_{rj} = g_{is} g_{js} = \delta_{ij}. \hfill (C.10)$$

In other words $g_{ij}$ is an orthogonal matrix. This can be proven formally as follows:

$$a_i = g_{ri} a'_r.$$
\[ a'_s = g_{sj}a_j \]

and

\[ a'_r = \delta_{rs}a'_s = \delta_{rs}g_{sj}a_j. \]

Therefore,

\[ a_i \equiv \delta_{ij}a_j = \frac{g_{ri}\delta_{rs}g_{sj}a_j}{g_{ri}g_{rj}} \]

and, for an arbitrary vector \( a_j \) we have:

\[ (\delta_{ij} - g_{ri}g_{rj})a_j = 0. \]

Consequently the bracketed term must vanish. This completes the proof.

### C.4 Transformation of stresses

Referring to the definition of traction vectors given in Section C.1, we have:

\[ T'_i = \sigma'_{ik}n'_k = \sigma'_{ik}g_{ks}n_s \]

and applying the transformation rule (C.8):

\[ T'_i = g_{ir}T_r = g_{ir}\sigma_{rs}n_s. \]

Therefore we have:

\[ (g_{ks}\sigma'_{ik} - g_{ir}\sigma_{rs})n_s = 0. \]

Since this equation holds for arbitrary \( n_s \), the bracketed term must vanish. Consequently:

\[ g_{ks}\sigma'_{ik} = g_{ir}\sigma_{rs} \]

multiplying by \( g_{js} \),

\[ g_{js}g_{ks}\sigma'_{ik} = g_{ir}g_{js}\sigma_{rs} \]

\[ \delta_{jk} \]
and using eq. (C.10), we have the transformation rule for stresses:

\[ \sigma'_{ij} = g_{ir}g_{js}\sigma_{rs}. \quad (C.11) \]

**Remark C.4.1** Denoting the stress tensor by \([\sigma]\) and \(g_{ij}\) by \([g]\), eq. (C.11) is the symmetric matrix triple product

\[ [\sigma'] = [g][\sigma][g]^T. \quad (C.12) \]