Appendix A

Definitions

The properties of norms, normed linear spaces, linear functionals and bilinear forms are listed in the following. The symbols $\alpha$ and $\beta$ denote real numbers. In Section A.6 the Legendre polynomials are defined and in Section A.8 the Schwarz inequality is derived.

A.1 Norms and seminorms

The norm of a function or vector is a measure of the size of the function or vector. Norms, are real nonnegative numbers defined on some space $X$. Norms, denoted by $\| \cdot \|_X$, have the following properties:

1. $\| u \|_X \geq 0$.
2. $\| u \|_X \neq 0$ if $u \neq 0$.
3. $\| \alpha u \|_X = |\alpha| \| u \|_X$.
4. $\| u + v \|_X \leq \| u \|_X + \| v \|_X$. This property is known as the triangle inequality.

As noted in Section 2.4.1, a familiar example of norms is the distance in Euclidean space. In connection with finite element analysis the commonly used norms are the maximum norm, defined in eq. (2.51); the $L_2$ norm, defined in eq. (2.52), and the energy norm, defined in eq. (2.49).

Seminorms satisfy properties 1, 3, and 4 of norms but do not satisfy property 2. Instead of property 2 seminorms have the property:

$$\| u\|_X = 0 \quad u \in \bar{X} \subset X, \quad u \neq 0.$$  

For example, $\| u \|_E$ defined by eq. (2.49) is a seminorm on the space defined by eq. (2.34) when $c = 0$. In this case $\bar{X}$ is the set of constant functions on the interval $(0, \ell)$. 

289
APPENDIX A. DEFINITIONS

A.2 Normed linear spaces:

A normed linear space $X$ is a family of elements $u, v, \ldots$ which have the following properties:

1. If $u \in X$ and $v \in X$ then $(u + v) \in X$.
2. If $u \in X$ then $\alpha u \in X$.
3. $u + v = v + u$
4. $u + (v + w) = (u + v) + w$
5. There is an unique element in $X$, denoted by $0$, such that $u + 0 = u$ for any $u \in X$.
6. Associated with every element $u \in X$ an unique element $-u \in X$ such that $u + (-u) = 0$.
7. $\alpha(u + v) = \alpha u + \alpha v$
8. $(\alpha + \beta)u = \alpha u + \beta u$
9. $\alpha(\beta u) = (\alpha \beta)u$
10. $1 \cdot u = u$
11. $0 \cdot u = 0$
12. Associated with every $u \in X$ is a real number $\|u\|_X$, called the norm. The norm has the properties listed in Section A.1.

A.3 Linear functionals

Let $X$ be a normed linear space and $F(v)$ a process which associates with every $v \in X$ a real number $F(v)$. $F(v)$ is called a linear functional or linear form on $X$ if it has the following properties:

1. $F(v_1 + v_2) = F(v_1) + F(v_2)$
2. $F(\alpha v) = \alpha F(v)$
3. $\|F(v)\| \leq C \|v\|_X$ with $C$ independent of $v$. The smallest possible value of $C$ is called the norm of $F$. 
A.4 Bilinear forms

Let $X$ and $Y$ be normed linear spaces and $B(u,v)$ a process that associates with every $u \in X$ and $v \in Y$ a real number $B(u,v)$. $B(u,v)$ is a bilinear form on $X \times Y$ if it has the following properties:

1. $B(u_1 + u_2, v) = B(u_1, v) + B(u_2, v)$
2. $B(u, v_1 + v_2) = B(u, v_1) + B(u, v_2)$
3. $B(\alpha u, v) = \alpha B(u, v)$
4. $B(u, \alpha v) = \alpha B(u, v)$
5. $|B(u,v)| \leq C \|u\|_X \|v\|_Y$ with $C$ independent of $u$ and $v$. The smallest possible value of $C$ is called the norm of $B$.

The space $X$ is called trial space and functions $u \in X$ are called trial functions. The space $Y$ is called test space and functions $v \in Y$ are called test functions. $B(u,v)$ is not necessarily symmetric.

A.5 Convergence

A sequence of functions $u_n \in X$ $(n = 1, 2, \ldots)$ converges in the space $X$ to the function $u \in X$ if for every $\epsilon > 0$ there is a number $n_\epsilon$ such that for any $n > n_\epsilon$ the following relationship holds:

$$\|u - u_n\|_X < \epsilon.$$  \hfill (A.1)

A.6 Legendre polynomials

The Legendre polynomials $P_n(x)$ are solutions of the Legendre differential equation for $n = 0, 1, 2, \ldots$:

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \quad -1 \leq x \leq 1.$$  \hfill (A.2)
The first eight Legendre polynomials are:

\begin{align*}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x) \\
P_6(x) &= \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5) \\
P_7(x) &= \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x).
\end{align*}

Legendre polynomials can be generated from the recursion formula:

\begin{equation}
(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x), \quad n = 1, 2, \ldots \tag{A.11}
\end{equation}

and Legendre polynomials satisfy the following relationship:

\begin{equation}
(2n + 1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x), \quad n = 1, 2, \ldots \tag{A.12}
\end{equation}

where the primes represent differentiation with respect to \(x\). Legendre polynomials satisfy the following orthogonality property:

\begin{equation}
\int_{-1}^{+1} P_i(x) P_j(x) \, dx = \begin{cases} 
\frac{2}{2i + 1} & \text{for } i = j \\
0 & \text{for } i \neq j.
\end{cases} \tag{A.13}
\end{equation}

All roots of Legendre polynomials are located in the interval \(-1 < x < +1\).

**A.7 Analytic functions**

**A.7.1 Analytic functions in \(\mathbb{R}^2\)**

A function \(f(x, y)\) is analytic on \(\bar{\Omega} \subseteq \mathbb{R}^2\) if for any \((x_0, y_0) \in \bar{\Omega}\) there is an \(r(x_0, y_0) > 0\) and a \(\kappa(x_0, y_0) > 0\) such that

\begin{equation}
f = \sum_{i, j=0}^{\infty} a_{ij}(x - x_0)^i(y - y_0)^j \quad \text{and} \quad \sum_{i, j=0}^{\infty} a_{ij}(i + j)!r^{i+j} \leq \kappa(x_0, y_0). \tag{A.14}
\end{equation}

From this definition it follows that if a solution is analytic then its derivatives are bounded. Specifically, for any \(s\):

\begin{equation}
\left| \frac{\partial^s u}{\partial x^k \partial y^{s-k}} \right| \leq s! \, r^s \kappa(x_0, y_0) \quad k = 0, 1, \ldots, s. \tag{A.15}
\end{equation}
A.7.2 Analytic curves in $\mathbb{R}^2$

A plane curve or arc is the set of points $\Gamma$ defined as follows:

$$\Gamma = \{(x, y) | x = x(t), \ y = y(t), \ t \in \bar{I} = [-1, 1]\}. \quad (A.16)$$

A plane curve $\Gamma$ is analytic if $x(t)$ and $y(t)$ are analytic functions of $t \in \bar{I}$ and

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 > 0 \ \text{for all} \ t \in \bar{I}. \quad (A.17)$$

A.8 The Schwarz inequality for integrals

**Definition A.8.1** The function $f(x)$ defined on the interval $a < x < b$ is square integrable if:

$$\int_a^b f^2 \, dx < \infty.$$  

**Theorem A.8.1** Let $f(x)$ and $g(x)$ be square integrable functions defined on the interval $a < x < b$. Then:

$$\left| \int_a^b fg \, dx \right| \leq \left( \int_a^b f^2 \, dx \right)^{1/2} \left( \int_a^b g^2 \, dx \right)^{1/2}.$$  

This is the Schwarz inequality for integrals. To prove this inequality we observe that:

$$\int_a^b (f + \lambda g)^2 \, dx \geq 0 \ \text{for any} \ \lambda$$

and therefore:

$$\int_a^b f^2 \, dx + 2\lambda \int_a^b fg \, dx + \lambda^2 \int_a^b g^2 \, dx \geq 0 \ \text{for any} \ \lambda. \quad (A.18)$$

On the left of this inequality is a quadratic expression for $\lambda$. To find the roots of this expression we need to compute:

$$\lambda = \frac{-\int_a^b fg \, dx \pm \sqrt{\left(\int_a^b fg \, dx\right)^2 - \int_a^b g^2 \, dx \int_a^b f^2 \, dx}}{\int_a^b g^2 \, dx}.$$  

Denoting the roots by $\lambda_1$ and $\lambda_2$ (A.18) can be written as: $(\lambda - \lambda_1)(\lambda - \lambda_2) \geq 0$.

We now observe that the roots cannot be real and simple because then we could select any $\lambda$ so that $\lambda_1 < \lambda < \lambda_2$ and we would have $(\lambda - \lambda_1)(\lambda - \lambda_2) < 0$. Therefore the radicand must be less than or equal to zero. This completes the proof.
Appendix B

Numerical quadrature

In the finite element method the terms of the coefficient matrices and right-hand side vectors are computed by numerical quadrature. Most commonly Gaussian quadrature is used, in some cases the Gauss-Lobatto quadrature is used. In one dimension the domain of integration is the standard element $I_{st}$. An integral expression on the standard element is approximated by a sum

$$\int_{-1}^{+1} f(\xi) \, d\xi \approx \sum_{i=1}^{n} w_i f(\xi_i) + R_n \quad (B.1)$$

where $w_i$ are the weights; $\xi_i$ are the abscissas and $R_n$ is the error term. The abscissas and weights are symmetric with respect to the point $\xi = 0$.

To evaluate an integral on other than the standard domain, the mapping function defined by eq. (2.56) is used for transforming the domain of integration to the standard domain. For example:

$$\int_{x_1}^{x_2} F(x) \, dx = \int_{-1}^{+1} F(Q(\xi)) \frac{x_2 - x_1}{2} \frac{d\xi}{f(\xi)} \quad \text{where} \quad Q(\xi) = \frac{1 - \xi}{2} x_1 + \frac{1 + \xi}{2} x_2.$$

B.1 Gaussian quadrature

In Gaussian quadrature the abscissa $x_i$ is the $i$th zero of Legendre polynomial $P_n$. The weights are computed from:

$$w_i = \frac{2}{(1 - x_i^2)|P_n'(x_i)|^2}. \quad (B.2)$$

\begin{footnotesize}
\footnote{Johann Carl Friedrich Gauss 1777-1855.}
\footnote{See, for example, Abramowitz, M. and Stegun, I., Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, U.S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series 55 (1964). A project is underway at the National Institute of Standards and Technology (the heir to National Bureau of Standards) to develop a replacement for this handbook that will be accessible through the Internet. See http://dlmf.nist.gov/about/.}
\end{footnotesize}
The abscissas and weights for Gaussian quadrature are listed in Table B.1 up to \( n = 8 \). The error term is:

\[
R_n = \frac{2^{2n+1} (2n+1)!}{(2n+1)!(2n)!} f^{(2n)}(\zeta) - 1 < \zeta < +1
\]

where \( f^{(2n)} \) is 2\(n\)th derivative of \( f \). It is seen from the error term that if \( f(\xi) \) is a polynomial of degree \( p \) and Gaussian quadrature is used then the integral will be exact (up to round-off errors) provided that \( n \geq (p + 1)/2 \). For example, to integrate a polynomial of degree 5, \( n = 3 \) is sufficient. For other than polynomial functions the rate of convergence depends on how well the integrand can be approximated by polynomials. It can be shown that if \( f(\xi) \) is a continuous function on \( I_{st} \) then the sum in eq. (B.1) converges to the true value of the integral.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \pm \xi_i )</th>
<th>( w_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.57735 02691 89626</td>
<td>1.00000 00000 00000</td>
</tr>
<tr>
<td>3</td>
<td>0.00000 00000 00000</td>
<td>0.88888 88888 88888</td>
</tr>
<tr>
<td></td>
<td>0.77459 66692 41483</td>
<td>0.55555 55555 55556</td>
</tr>
<tr>
<td>4</td>
<td>0.33998 10435 84856</td>
<td>0.65214 51548 62546</td>
</tr>
<tr>
<td></td>
<td>0.86113 63115 94053</td>
<td>0.34785 48451 37454</td>
</tr>
<tr>
<td>5</td>
<td>0.00000 00000 00000</td>
<td>0.56888 88888 88889</td>
</tr>
<tr>
<td></td>
<td>0.53846 93101 05683</td>
<td>0.47862 86704 99366</td>
</tr>
<tr>
<td></td>
<td>0.90617 98459 36664</td>
<td>0.23692 68850 56189</td>
</tr>
<tr>
<td>6</td>
<td>0.23861 91860 83197</td>
<td>0.46791 39345 72691</td>
</tr>
<tr>
<td></td>
<td>0.66120 93864 66265</td>
<td>0.36076 15730 48139</td>
</tr>
<tr>
<td></td>
<td>0.93246 95142 03152</td>
<td>0.17132 44923 79170</td>
</tr>
<tr>
<td>7</td>
<td>0.00000 00000 00000</td>
<td>0.41795 01836 73469</td>
</tr>
<tr>
<td></td>
<td>0.40584 51513 77397</td>
<td>0.38183 00505 05119</td>
</tr>
<tr>
<td></td>
<td>0.74153 11855 99394</td>
<td>0.27970 53914 89277</td>
</tr>
<tr>
<td></td>
<td>0.94910 79123 42759</td>
<td>0.12948 49661 68870</td>
</tr>
<tr>
<td>8</td>
<td>0.18343 46424 95650</td>
<td>0.36268 37833 78362</td>
</tr>
<tr>
<td></td>
<td>0.52553 24099 16329</td>
<td>0.31370 66458 77887</td>
</tr>
<tr>
<td></td>
<td>0.79666 47741 3627</td>
<td>0.22238 10344 53374</td>
</tr>
<tr>
<td></td>
<td>0.96028 98564 97536</td>
<td>0.10122 85362 90376</td>
</tr>
</tbody>
</table>

The integration procedure can be extended directly to the standard quadrilateral element and the standard hexahedral element. For the standard quadrilateral element:

\[
\int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta) \, d\xi \, d\eta = \sum_{i=1}^{n_\xi} \sum_{j=1}^{n_\eta} w_i w_j f(\xi_i, \eta_j) \tag{B.3}
\]

where \( n_\xi \) (resp. \( n_\eta \)) is the number of quadrature points along the \( \xi \) (resp. \( \eta \) axis). Usually but not necessarily \( n_\xi = n_\eta \) is used. Analogously, for the standard
B.2. GAUSS-LOBATTO QUADRATURE

hexahedral element:

\[ \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta, \zeta) \, d\xi d\eta d\zeta = \sum_{i=1}^{n_\xi} \sum_{j=1}^{n_\eta} \sum_{k=1}^{n_\zeta} w_i w_j w_k f(\xi_i, \eta_j, \zeta_k). \] (B.4)

B.2 Gauss-Lobatto quadrature

In the Gauss-Lobatto quadrature the abscissas are as follows: \( x_1 = -1, x_n = 1 \) and for \( i = 2, 3, \ldots, n - 1 \) the \((i-1)\)st zero of \( P_n'(x) \). The weights are:

\[ w_i = \begin{cases} \frac{2}{n(n-1)} & \text{for } i = 1 \text{ and } i = n, \\ \frac{2}{n(n-1)|P_{n-1}'(x_i)|^2} & \text{for } i = 2, 3, \ldots, (n-1). \end{cases} \] (B.5)

The abscissas and weights for Gauss-Lobatto quadrature are listed in Table B.2 up to \( n = 8 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \pm \xi_i )</th>
<th>( w_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.00000 00000 00000</td>
<td>1.00000 00000 00000</td>
</tr>
<tr>
<td>3</td>
<td>0.00000 00000 00000</td>
<td>1.33333 33333 33333</td>
</tr>
<tr>
<td></td>
<td>1.00000 00000 00000</td>
<td>0.33333 33333 33333</td>
</tr>
<tr>
<td>4</td>
<td>0.44721 35954 99999</td>
<td>0.83333 33333 33333</td>
</tr>
<tr>
<td></td>
<td>0.00000 00000 00000</td>
<td>0.16666 66666 66667</td>
</tr>
<tr>
<td>5</td>
<td>0.00000 00000 00000</td>
<td>0.71111 11111 11111</td>
</tr>
<tr>
<td></td>
<td>0.05465 36707 07977</td>
<td>0.54444 44444 44444</td>
</tr>
<tr>
<td></td>
<td>0.00000 00000 00000</td>
<td>0.10000 00000 00000</td>
</tr>
<tr>
<td>6</td>
<td>0.28523 15164 80645</td>
<td>0.55485 83703 54386</td>
</tr>
<tr>
<td></td>
<td>0.76505 53239 29465</td>
<td>0.37847 49562 97847</td>
</tr>
<tr>
<td></td>
<td>0.00000 00000 00000</td>
<td>0.06666 66666 66667</td>
</tr>
<tr>
<td>7</td>
<td>0.00000 00000 00000</td>
<td>0.48761 90476 19048</td>
</tr>
<tr>
<td></td>
<td>0.46884 87934 70714</td>
<td>0.43174 53812 09863</td>
</tr>
<tr>
<td></td>
<td>0.83022 38962 78567</td>
<td>0.27682 60473 61566</td>
</tr>
<tr>
<td></td>
<td>0.00000 00000 00000</td>
<td>0.04761 90476 19048</td>
</tr>
<tr>
<td>8</td>
<td>0.20929 92179 02479</td>
<td>0.41245 87946 58704</td>
</tr>
<tr>
<td></td>
<td>0.59170 01814 33142</td>
<td>0.34112 26924 83504</td>
</tr>
<tr>
<td></td>
<td>0.87174 01485 09607</td>
<td>0.21070 42271 43506</td>
</tr>
<tr>
<td></td>
<td>1.00000 00000 00000</td>
<td>0.03571 42857 14286</td>
</tr>
</tbody>
</table>

The error term is:

\[ R_n = \frac{-n(n-1)^3 2^{2n-1} [(n-2)!]^4}{(2n-1)[(2n-2)!]^3} f^{(2n-2)}(\zeta) \quad \text{for } -1 < \zeta < +1. \]
from which it follows that if $f(\xi)$ is a polynomial of degree $p$ and Gauss-Lobatto quadrature is used then the integral will be exact (up to round-off errors) provided that $n \geq (p + 3)/2$. 