Homework 4
Elliptic Problems

1

Consider the following BVP: Find the function \( u(x, y) \) for \( x \in [-1, 1] \) \( y \in [-1, 1] \) that satisfies

\[
-\left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 1000
\]

subject to the boundary conditions

\[
u(-1, y) = u(1, y) = u(x, -1) = u(x, 1) = 0
\]

\( a) \) Use the FEM in the COMSOL program, select appropriate element basis functions and obtain and approximate solution of the problem. Verify the reliability of the approximate solution by comparing your results against the given exact solution AND also by performing a mesh extension study.

\( b) \) Use the FEM in the Ansys or Abaqus programs, select appropriate element basis functions and obtain and approximate solution of the problem. Verify the reliability of the approximate solution by comparing your results against the given exact solution AND also by performing a mesh extension study.

2

Consider the following BVP: Find the function \( u(x, y) \) for \( x \in [0, a] \) \( y \in [0, b] \) that satisfies

\[
-\left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 1000
\]
subject to the boundary conditions

\[ u(0, y) = u(a, y) = 0 \]

\[ \frac{\partial u}{\partial y}(x, 0) = \frac{\partial u}{\partial y}(x, b) = 0 \]

a) Assume \( a = b = 1 \) and use the FEM in the COMSOL program, select appropriate element basis functions and obtain an approximate solution of the problem. Verify the reliability of the approximate solution by comparing your results against the given exact solution AND also by performing a mesh extension study.

b) Use the FEM in the Ansys or Abaqus programs, select appropriate element basis functions and obtain an approximate solution of the problem. Verify the reliability of the approximate solution by comparing your results against the given exact solution AND also by performing a mesh extension study.

3

The function \( u(x, y) = x^2 - y^2 \) satisfies the elliptic partial differential equation

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

and also the boundary conditions

\[ u(0, y) = -y^2 \]
\[ u(x, 0) = x^2 \]
\[ u(x, 1) = x^2 - 1 \]
\[ u(1, y) = 1 - y^2 \]

for \( x \in [0, 1] \) and \( y \in [0, 1] \). It can be used to model the two-dimensional steady potential flow of an ideal fluid near a stagnation point by taking it as the potential of the flow such that the velocity vector \( \mathbf{v} \) is

\[ \mathbf{v} = \nabla u \]

i.e. \( v_x = \partial u / \partial x \) and \( v_y = \partial u / \partial y \).
Closely associated to this problem is the (complex conjugate) stream function $\psi = 2xy$ related to the velocity components by $v_x = \partial \psi / \partial y$ and $v_y = -\partial \psi / \partial x$. The stream function also satisfies Laplace’s equation subject to the boundary conditions

$$
\psi(x, 0) = 0 \\
\psi(0, y) = 0 \\
\psi(x, 1) = 2x \\
\psi(1, y) = 2y
$$

a) Show that the given functions does satisfy Laplace’s equation as well as all the boundary conditions.

b) Use the FEM in the COMSOL program with $P1$ basis functions and obtain and approximate solution of the problem. Verify the reliability of the approximate solution by comparing your results against the given exact solution AND also by performing a mesh extension study.

c) Repeat the calculation using $P2$ and $P3$ basis functions.

d) Use the FEM in the Ansys or Abaqus programs, identify and select element basis functions and obtain and approximate solution of the problem. Verify the reliability of the approximate solution by comparing your results against the given exact solution AND also by performing a mesh extension study.

4

Consider the following BVP: Find the function $u(r, z)$ for $r \in [0, 1]$ $z \in [0, 1]$ that satisfies

$$
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0
$$

subject to the boundary conditions

$$
u(r, 0) = u(1, z) = 0$$

$$u(r, 1) = 1$$
and \( u(0, z) \) bounded.

The problem could be a model to estimate the electrical potential \( u(r, z) \) in a cylinder exposed to zero (ground) potential at its base and side surfaces and a constant potential of 1 at the top face.

**a)** Use the FEM in the COMSOL program, select appropriate element basis functions and obtain and approximate solution of the problem. Verify the reliability of the solution.

**b)** Use the FEM in the Ansys or Abaqus program, select appropriate element basis functions and obtain and approximate solution of the problem. Verify the reliability of the solution.

Hint: The exact solution of this problem is readily obtained by separation of variables and is given by

\[
    u(r, z) = \sum_{n=1}^{\infty} c_n J_0(k_n r) \sinh(k_n z)
\]

where \( J_0 \) is the Bessel function of first kind order zero. the Fourier-Bessel coefficients are given by

\[
    c_n = \frac{1}{\sinh(k_n)} \frac{\int_0^1 r J_0(k_n r) dr}{\int_0^1 r J_2^2(k_n r) dr} = \frac{2 \int_0^1 r J_0(k_n r) dr}{\sinh(k_n) J_1^2(k_n)}
\]

and the eigenvalues \( k_n \) are the roots of

\[
    J_0(k_n) = 0
\]

A square plate is made of steel \((E = 2 \times 10^{11} \, N/m^2, \nu = 0.3)\) and has width = length = 1 m and thickness = 0.025 m and is firmly clamped around its edges. A uniform pressure \( p_0 = 10 \times 10^6 \, Pa \) acts normal to the plane of the plate.

**a)** Use the FEM in the COMSOL program, select appropriate element basis functions and compute an approximation to the resulting deflection \( u(x, y, z) \). Verify the validity of the solution.
b) Use the FEM in the Ansys or Abaqus programs, select appropriate element basis functions and compute an approximation to the resulting deflection $u(x, y, z)$. Verify the reliability of the solution.