Numerical Methods in Laminar and Turbulent Flow

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PRESSURE METHODS FOR THE APPROXIMATE SOLUTION OF THE NAVIER-STOKES EQUATIONS

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SUMMARY. In this paper we will develop and apply a family of powerful finite difference methods for the numerical solution of the Navier-Stokes equations for incompressible fluids. The power is derived by combining advantageous aspects of the Marker-and-Cell (MAC) method with several special techniques for hyperbolic and parabolic equations. The stability of the methods, in the discrete $L_2$ norm, is demonstrated to be independent of the pressure and is shown to depend only on the discretizations chosen for the convective and viscous terms. Appropriate choices of such discretizations are then suggested. A numerical example is described and discussed.

1. INTRODUCTION.

The partial differential equations governing the laminar flow of an incompressible fluid are obtained from the physical principles of conservation of linear momentum and volume. Such equations, in their primitive variables, and in two space dimensions, can be written as

\[
\begin{align*}
\frac{3u}{3t} + \frac{3u}{3x} + \frac{3v}{3y} &= - \frac{3p}{3x} + \frac{\nu}{3x} \left( \frac{3u}{3x} + \frac{3u}{3y} \right) \\
\frac{3v}{3t} + \frac{3v}{3x} + \frac{3v}{3y} &= - \frac{3p}{3y} + \frac{\nu}{3y} \left( \frac{3v}{3x} + \frac{3v}{3y} \right) \\
\frac{3u}{3x} + \frac{3v}{3y} &= 0,
\end{align*}
\]

(1)

where $u(x,y,t)$ and $v(x,y,t)$ are the velocity components in the $x$ and $y$ directions, respectively. The normalized pressure $p(x,y,t)$ is defined as the ratio of pressure to density, where the density is assumed to be constant. The
kinematic viscosity coefficient \( \nu \) is assumed to be constant and nonnegative.

If \( \Omega \subset \mathbb{R}^2 \) denotes the spatial domain for equations (1), and \( \Gamma \) is the boundary of \( \Omega \), the following initial and boundary conditions are associated with equations (1):

\[
\begin{align*}
\begin{cases}
    u(x,y,0) = u_0(x,y), & (x,y) \in \Omega \\
    v(x,y,0) = v_0(x,y), & (x,y) \in \Omega \\
    u(x,y,t) = u_b(x,y,t), & (x,y) \in \Gamma, \\0 < t < T \\
    v(x,y,t) = v_b(x,y,t) \\
\end{cases}
\end{align*}
\]

where \( u_0(x,y), v_0(x,y), u_b(x,y,t) \) and \( v_b(x,y,t) \) are given.

An analytical solution of equations (1) would yield values for the unknowns \( u, v \) and \( p \) at each instant in time and at every point in the flow field. However, except for very few particular cases, an analytical solution can rarely be constructed and a numerical, computer oriented technique must be introduced.

2. SEMIDISCRETE FORMULATION.

In order to solve equations (1) numerically, we introduce, as in the MAC method [3], a finite difference mesh which consists of rectangular cells of width \( \Delta x \) and height \( \Delta y \) which cover \( \Omega \). The discrete pressure and fluid velocities are located at the cell positions shown in Fig. 1: the horizontal velocity \( u \) in the \( x \)-direction is defined at the center of each vertical side of a cell, the vertical velocity \( v \) in the \( y \)-direction is defined at the center of each horizontal side, and the pressure \( p \) is defined at each cell center. Each cell is numbered at its center with indices \( i \) and \( j \), where \( i \) is the cell's column number in the \( x \)-direction from left to right, and \( j \) is the cell's row number in the \( z \)-direction from bottom to top.

![Fig. 1: Position of field variables.](image)

If equations (1) are spatially discretized via the conventional finite difference method by using centered differences for the \( \nu \) terms and for the continuity equation, there results the following matrix system of coupled first order differential equations:

\[
\begin{align*}
    \frac{du_{i+1/2,j}}{dt} &= -F_{i+1/2,j} - \frac{P_{i+1,j} - P_{i,j}}{\Delta x} \\
    \frac{dv_{i,j+1/2}}{dt} &= -G_{i,j+1/2} - \frac{P_{i,j+1} - P_{i,j}}{\Delta y} \\
    \frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x} + \frac{v_{i,j+1/2} - v_{i,j-1/2}}{\Delta y} &= 0,
\end{align*}
\]

where \( F_{i+1/2,j} \) and \( G_{i,j+1/2} \) contain the finite differences corresponding to the spatial discretization of the convective and viscous terms.

Once a discretization for the convective and for the viscous terms has been chosen, \( F_{i+1/2,j} \) and \( G_{i,j+1/2} \) are determined and the differential equations in (4) are readily solvable numerically with any one of the several finite difference methods for ordinary differential equations which are abundantly available. The selected method can be one step or multistep and explicit or implicit with respect to \( u_{i+1/2,j} \) and \( v_{i,j+1/2} \).

3. EXPLICIT METHODS.

A very simple way to solve system (4) is to discretize the ordinary differential equations of (4) with the explicit Euler scheme as follows:

\[
\begin{align*}
    \begin{cases}
    u_{i+1}^{n+1} - u_{i}^{n} &= -F_{i+1/2,j} - \frac{P_{i+1,j}^{n} - P_{i,j}^{n}}{\Delta x} \\
    v_{i,j+1}^{n+1} - v_{i,j}^{n} &= -G_{i,j+1/2} - \frac{P_{i,j+1}^{n} - P_{i,j}^{n}}{\Delta y} \\
    \end{cases}
\end{align*}
\]

and to discretize the continuity equation, which we require to be satisfied at each time step, as follows:

\[
\begin{align*}
    \begin{cases}
    u_{i+1/2,j}^{n+1} - u_{i-1/2,j}^{n+1} + \frac{v_{i,j+1/2}^{n+1} - v_{i,j-1/2}^{n+1}}{\Delta x} &= 0 \\
    \end{cases}
\end{align*}
\]

In (5) and (6), the superscript \( n \) or \( n+1 \) indicates the time at which the variable is to be evaluated, and \( \Delta t \) denotes
the time step. Thus, for example, \( u^{n+1}_{i+1,j} \) is to be evaluated at time \( n \Delta t \).

Equations (5)-(6), at each time step, constitute a linear system of equations with unknowns \( u^{n+1}_{i,j} \), \( v^{n+1}_{i,j} \), and \( p^{n+1}_{i,j} \). However, the order of this system can be reduced by substituting expressions for \( u^{n+1} \) and \( v^{n+1} \) from (5) into (6) to yield

\[
\frac{p^{n+1}_{i,j}}{\Delta t} = \frac{\Delta}{\partial x} \left( u^{n}_{i,j+1} - u^{n}_{i,j} \right) + \frac{\Delta}{\partial y} \left( v^{n}_{i,j+1} - v^{n}_{i,j} \right) - \frac{1}{\Delta t} \left( p^{n}_{i+1,j} - p^{n}_{i,j} \right) + \frac{1}{\Delta t} \left( p^{n}_{i,j+1} - p^{n}_{i,j} \right)
\]

(7)

On the cells, the set of equations (7) constitutes a linear system with unknowns \( p^{n+1}_{i,j} \). Since the velocity, not the pressure, is given as a boundary condition on \( \Gamma \), the matrix of coefficients for the linear system of equations (7) is singular and hence the pressure field \( p^{n+1}_{i,j} \) is determined up to an arbitrary additive constant which cancels in (5).

Thus, the velocity field \( u^{n+1}_{i,j} \) and \( v^{n+1}_{i,j} \) is determined uniquely. In practice, linear system (7) can be solved iteratively by using the successive overrelaxation iterative method in a very efficient way to yield, simultaneously, \( u^{n+1} \), \( v^{n+1} \) and \( p^{n+1} \) (see, e.g., [2]).

In all previous considerations the structure of the terms \( F^{n}_{i+1,j} \) and \( G^{n}_{i,j+1} \) has been left out. However, the structure of such terms plays a very important role for the consistency, the accuracy and for the stability of the numerical method (5), (6). Note that, since \( F^{n}_{i+1,j} \) and \( G^{n}_{i,j+1} \) contain the finite differences corresponding to the spatial discretization of the convective and viscous terms of the Navier-Stokes equations (1), an appropriate choice of \( F^{n}_{i+1,j} \) and \( G^{n}_{i,j+1} \) is implied from considerations of the parabolic (hyperbolic if \( \nu = 0 \)) equations that one obtains from the momentum equations in (1) by neglect of the pressure terms.

Let us give, as examples, two different expressions to \( F^{n}_{i+1,j} \) and \( G^{n}_{i,j+1} \) and, as a result, derive two different methods with different properties for the numerical solution of (1).

First assume that the viscosity \( \nu \) is strictly positive. In this case, by use of centered finite differences to discretize both the convective and the viscous terms, second order accurate formulas for \( F^{n}_{i+1,j} \) and \( G^{n}_{i,j+1} \) are

\[
\begin{align*}
\frac{u^{n+1}_{i+1,j} - u^{n+1}_{i,j}}{\Delta x} + \frac{v^{n+1}_{i+1,j} - v^{n+1}_{i,j}}{\Delta y} &= u^{n}_{i+1,j} - u^{n}_{i,j} + \frac{1}{\Delta t} \left( p^{n}_{i+1,j} - p^{n}_{i,j} \right) + \frac{1}{\Delta t} \left( p^{n}_{i,j+1} - p^{n}_{i,j} \right) \\
\frac{u^{n+1}_{i+1,j} - u^{n+1}_{i,j}}{\Delta x} &= u^{n}_{i+1,j} - u^{n}_{i,j} + \frac{1}{\Delta t} \left( p^{n}_{i+1,j} - p^{n}_{i,j} \right) + \frac{1}{\Delta t} \left( p^{n}_{i,j+1} - p^{n}_{i,j} \right)
\end{align*}
\]

(8)

It will be shown later that, if the viscosity is zero and \( F^{n}_{i+1,j} \), \( G^{n}_{i,j+1} \) are given by (8), the numerical method (5), (6) is always unstable.

Finite difference formulas for \( F^{n}_{i+1,j} \) and \( G^{n}_{i,j+1} \) which are only first order accurate but applicable for any nonnegative viscosity \( \nu \) are given as follows:

\[
\begin{align*}
\frac{u^{n+1}_{i+1,j} - u^{n+1}_{i,j}}{\Delta x} &= u^{n}_{i+1,j} - u^{n}_{i,j} + \frac{1}{\Delta t} \left( p^{n}_{i+1,j} - p^{n}_{i,j} \right) + \frac{1}{\Delta t} \left( p^{n}_{i,j+1} - p^{n}_{i,j} \right) \\
\frac{u^{n+1}_{i+1,j} - u^{n+1}_{i,j}}{\Delta x} &= u^{n}_{i+1,j} - u^{n}_{i,j} + \frac{1}{\Delta t} \left( p^{n}_{i+1,j} - p^{n}_{i,j} \right) + \frac{1}{\Delta t} \left( p^{n}_{i,j+1} - p^{n}_{i,j} \right)
\end{align*}
\]

(9)

In (9) the upper or lower line of each brace is to apply as the corresponding coefficient is nonnegative or not.

The discrete variables \( u^{n}_{i,j+1} \) and \( v^{n}_{i+1,j} \) in (8) and
(9) are not defined (see Fig. 1), thus a simple average, using the two closest scalar grid points, is to be used.

Formulas for $p^n_{i,j}$ and $C^n_{i,j}$ other than (8) and (9) can be given (see e.g. [2]).

4. STABILITY CONSIDERATIONS

Once a space mesh has been chosen, the choice of the time increment and, at times, even the space steps have to satisfy special conditions for stability.

Consider the explicit difference scheme (5), (6). In matrix notation, (5) can be written in the following way:

$$\frac{w^{n+1} - w^n}{\Delta t} = -H(w^n)w^n - A^{-1}p^{n+1},$$

where $w^n$ denotes a vector whose elements are the velocities $u^n_{i,j}$ and $v^n_{i,j}$ in each cell. $H(w^n)$ is a square matrix whose structure depends on the formulas used in $p^n_{i,j}$ and $C^n_{i,j}$. $p^{n+1}$ denotes a vector whose elements are the pressure values $p^{n+1}_{i,j}$ in each cell, and rectangular matrix $A$ is the matrix analogue of the finite difference gradient.

The discrete continuity equation (6) in matrix notation is

$$A^T w^{n+1} = 0.$$

The system of finite difference equations (7) for the pressure in matrix notation is derived by combining (10) with (11):

$$A^T A^{-1} p^{n+1} = \frac{1}{\Delta t} A^T w^n - A^T H(w^n) w^n.$$

Assuming the pressure to be given at, at least, one boundary point, the matrix $A^T A$ is nonsingular and a solution for $p^{n+1}$ can be expressed as follows:

$$p^{n+1} = (A^T A)^{-1} A^T \left[ I - \frac{1}{\Delta t} H(w^n) \right] w^n.$$

Substitution of (13) into (10) then yields

$$w^{n+1} = [I - A(A^T A)^{-1} A^T] [I - (\Delta t) H(w^n)] w^n.$$

To determine the condition under which (14) is stable, consider the discrete $L^2$ norm of $w^{n+1}$:

$$\| w^{n+1} \|_2^2 \leq \| I - A(A^T A)^{-1} A^T \|_2^2 \| w^n \|_2^2 - (\Delta t) \| H(w^n) \|_2^2 \| w^{n+1} \|_2^2.$$

Note now that since the matrix $[I - A(A^T A)^{-1} A^T]$ is a projector, its $L^2$ norm is unity and hence (15) implies that (14) is stable provided the following inequality is satisfied at each time step:

$$\| I - (\Delta t) H(w^n) \|_2^2 \leq 1.$$

The stability condition (16) is independent of the pressure. It depends only on the structure of the matrix $H(w^n)$, that is, on the expressions chosen for $p^n_{i,j}$ and $C^n_{i,j}$. In particular, if $p^n_{i,j}$ and $C^n_{i,j}$ are given by (8) then (16) is satisfied if

$$\left\{ \begin{align*}
\frac{\Delta x}{\Delta y} \max_{i,j} |u^n_{i,j}| & \leq 2v \\
\frac{\Delta y}{\Delta x} \max_{i,j} |v^n_{i,j}| & \leq 2v \\
2v (\Delta t) \left[ \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right] & \leq 1.
\end{align*} \right.$$

If $p^n_{i,j}$ and $C^n_{i,j}$ are given by (9), then (16) is satisfied if

$$\left\{ \begin{align*}
\frac{\Delta x}{\Delta y} \max_{i,j} \left( \frac{|u^n_{i,j}|}{\Delta x} + \frac{|v^n_{i,j}|}{\Delta y} + 2v \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) & \leq 1.
\end{align*} \right.$$

In general, if one uses different formulas for $p^n_{i,j}$ and $C^n_{i,j}$, the matrix $H(w^n)$ is different and the stability restrictions on the finite difference steps are obtained by requiring that (16) be satisfied.

5. IMPLICIT METHODS.

The use of implicit numerical methods for systems of evolution equations is usually recommended when large time steps are essential. The price one pays for this capability is that one has to solve, at each time step, a system of algebraic or transcendental equations involving all the dependent variables.

A simple implicit method for the Navier-Stokes equations (1) is obtained by discretizing the differential equations in (4) with a modified implicit Euler scheme. Such a method, in matrix notation, can be written as

$$\frac{w^{n+1} - w^n}{\Delta t} = -H(w^n)w^{n+1} - A^{-1}p^{n+1},$$

$$A^T w^{n+1} = 0,$$
where, again, the structure of the matrix $H(W^n)$ depends upon the discretization chosen for the convective and viscous terms. Note that system (19)-(20), though implicit, is linear with respect to $w^{n+1}$ and $p^{n+1}$.

With regard to the stability of the implicit formulas (19)-(20), assume that $H(W^n)$ is chosen in such a fashion that $[I + (\Delta t)H(W^n)]$ is nonsingular. Thus (19) implies

$$w^{n+1} = [I + (\Delta t)H(W^n)]^{-1} w^n - (\Delta t)Ap^{n+1}.$$  

Substitution of (21) into (20) yields a finite difference system of equations for the pressure:

$$\Delta t A^T [I + (\Delta t)H(W^n)]^{-1} Ap^{n+1} = A^T [I + (\Delta t)H(W^n)]^{-1} w^n.$$  

Assuming the pressure to be known at, at least, one boundary point, system (22) has a unique solution $p^{n+1}$ provided $\Delta t$ is sufficiently small. In practice, this sufficient condition appears to be unnecessarily restrictive [1]. Thus, the solution $p^{n+1}$ of (22) is given by

$$p^{n+1} = \frac{1}{\Delta t} (A^T R^n A)^{-1} A^T R^n w^n,$$

where $R^n$ denotes the matrix $[I + (\Delta t)H(W^n)]^{-1}$. Substitution of (23) into (21) yields

$$w^{n+1} = R^n [I - A(A^T R^n A)^{-1} A^T R^n] w^n.$$

Note now that since $[I - A(A^T R^n A)^{-1} A^T R^n]$ is a projector, (24) implies

$$\|w^{n+1}\|_2 \leq \|R^n\|_2 \|w^n\|_2,$$

so that (24) is stable if the following inequality is satisfied

$$\|I + (\Delta t)H(W^n)\|_2^{-1} \leq 1.$$  

Thus, one can observe that the stability condition (26) is independent of the pressure. It depends only on the structure of the matrix $H(W^n)$, that is, on the discretization chosen for the convective and viscous terms. Moreover, since (26) is also the stability condition for the implicit scheme that one obtains from (19) by neglecting the pressure term, we can state that any stable implicit discretization for the parabolic (hyperbolic if $\nu = 0$) equations that one obtains from the momentum equations in (1) by neglecting the pressure gradient terms can be adapted to obtain a stable method of the form (19)-(20) for the complete Navier-Stokes equations (1).

The previous considerations also apply to the Navier-Stokes equations in three space dimensions (see, e.g., [2]).

6. A DRIVEN CAVITY CALCULATION.

As an application of the methods discussed above, consider the familiar driven cavity problem. Let us follow the flow which develops in a square cavity whose sides have length 2. Assume that the fluid has viscosity $\nu = 0.4$ and that at the initial time $t = 0$ the velocities are

$$u_0(x,y) = v_0(x,y) = 0.$$  

In addition, the bottom and side boundaries are fixed, while the top boundary is assumed to be moving with constant velocity $u_a = 0.5$. The discretization parameters are taken to be $Ax = Ay = 0.2$ and $\Delta t = 0.01$.

Figures 2, 3, 4 and 5 show the computed solution obtained at times $t = 0.1, t = 0.2, t = 0.3$ and $t = 1$, respectively, by using the explicit method (5), (6) with the choice (9) for the terms $F^n_{i+\frac{1}{2},j\pm\frac{1}{2}}$ and $G^n_{i, j\pm\frac{1}{2}}$. The numerical solution shown in Fig. 5 represents the steady solution for this problem.

Because the maximum velocity in this cavity flow problem is at the top, moving boundary, our choice of space and time steps is consistent with stability condition (18).

REFERENCES


