Min-Max Load Model for Optimizing Machining Fixture Performance

Spherical-tipped locators and clamps are often used for the restraint of castings during machining. For structurally rigid castings, contact region deformation and micro-slippage are the predominant modes of workpiece displacement. In turn, contact region deformation and micro-slippage are heavily influenced by contact region loading. Maximum loading magnitude has been shown to be a good indicator of workpiece displacement. This paper presents an algorithm which uses the min-max loading criteria as a basis for determining the optimal layout of locators and clamps as well as clamp actuation intensities. The experimentation used for model validation is also discussed along with experimental results.

1 Introduction

Spherical-tipped locators and clamps are often used for the restraint of castings during machining. Two critical parameters in fixture design are locator and clamp placement and clamp actuation intensity. When selecting these parameters, the designer's primary concern is workpiece displacement.

For structurally rigid castings, contact region deformation and micro-slippage are the predominant modes of workpiece displacement. Researchers such as Lee and Haynes (1987) and Daimon et al. (1985) have adopted contact mechanics models to predict the impact of locator and clamp placement and clamping intensity on workpiece displacement. In general, these techniques consider the effect of static loading on quasi-static workpiece displacement. They assume a workpiece to be an isotropic elastic body in frictional contact with rigid elements. The partial differential equations defining stress and elastic strain are linearized through the finite element method (FEM).

Boundary conditions are created for the nodes in contact with the fixture elements. These nodes are assumed to be restrained by static friction. When the model is solved, the forces exerted on these nodes are checked against Coulombic friction constraints. If violated, a new set of boundary constraints are created which reflect frictional sliding. The model is re-solved, and the frictional constraints re-evaluated. This cycle is repeated until a solution which satisfies all boundary constraints is found.

Researchers such as Mcass and De Vries (1991, 1990) and Reckert et al. (1993) have extended these models into nonlinear optimization algorithms for determining the locations of locators and clamps and actuation intensities for minimizing workpiece displacement.

These algorithms start with a feasible design and iteratively improve it until a stopping condition is met. These conditions are typically the achievement of a local optimum solution, the surpassment of an iteration limit, or the failure to improve the solution by a required minimum amount. Since the models are large in size and require the remeshing of the FEM at every iteration, each iteration is computationally expensive. Consequently it is advantageous to start with a good initial feasible design. To obtain these initial designs, the fixture designer would prefer to use a far less expensive design optimization algorithm.

For example, since deformation and micro-slippage are a direct result of contact region loading, it is reasonable to assume that a design which results in less loading throughout a series of cuts will also result in less workpiece displacement. This correlation has been shown through a series of simulation experiments (De Meter, 1994b). Likewise the minimum contact region loads necessary to keep a workpiece in static equilibrium can be predicted using a small, linear programming model. Consequently, this model could be used as the basis of a simple, but effective design optimization algorithm.

In addition to displacement minimization, the designer must also assure that two other functional requirements are met. The first is that the locators provide the proper kinematic restraint. The second is that the locators and clamps provide total restraint in order to assure that the workpiece is not displaced by small, random forces prior to machining. Validation techniques for these requirements appear in the literature. Examples include those of Salisbury and Roth (1983), Assada and Kitwaga (1989), Chou et al. (1989), and De Meter (1994a). Unfortunately little progress has been made toward the integration of both requirements into a general design optimization procedure.

This paper describes an algorithm for determining locator and clamp positions and clamp actuation intensities that result in minimum loading throughout a series of cuts, and
which satisfy kinematic restraint and total restraint requirements. This algorithm is based on the min-max load model described by De Meter (1994b), in which the workpiece is treated as a rigid body subject to Coulombic friction and a continuum of external loads.

In the following section, the problem and its assumptions are described in detail. Subsequently the general model is developed and the design optimization algorithm is described. This is followed by a description of the computational experiments used to validate the effectiveness of the algorithm.

2 Problem Definition

Multiple machining operations are to be performed on a prismatic workpiece. The workpiece is restrained by a fixture with L locators and C power clamps. Both the locators and clamp elements have spherical tips. The workpiece and the fixture elements are considered rigid and thus all contact regions are points lying on the workpiece surface. Friction exists at all contact regions. The clamps are singly actuated mechanisms.

The loads at the clamp contact regions are assumed to be dictated by the clamp actuation intensities. During clamping and machining, the loads at the locator contact regions are assumed to be distributed as uniformly as possible. Portions of selected workpiece surfaces are to be used for location or clamping. The locators and clamps are to be positioned within these regions.

The designer’s objective is to choose the positions of the fixture elements and the clamp actuation intensities so that the loads at the contact regions are minimized throughout all operations. Additional requirements are that the fixture configuration provide complete kinematic restraint and total restraint.

3 Model Development

In this section, the mathematical constraints for all relevant phenomena are developed. Contact region wrench constraints are described next.

3.1 Contact Region Wrench Constraints. Each contact region is a point which lies on the workpiece surface. This is illustrated in Fig. 1. Let \( p_j \) be the position of contact region \( j \), \( \eta_j \) the surface normal, \( \mu_j \) the coefficient of static friction, and \( f_j \) the resultant force exerted by the region on the workpiece. By definition \( f_j \) must pass through \( p_j \) and lie within a friction cone whose axis is parallel to \( \eta_j \), and whose half-angle, \( \theta_j \), is defined by:

\[
\theta_j = \tan^{-1} \mu_j
\]

(1)

Note that \( \beta \) represents the tangent plane at \( p_j \).

\( f_j \) can be modeled conservatively with a polyhedral cone approximation as shown in Fig. 2. Let \( f_{ji} \) be the \( i \)th spanning force of the polyhedral cone approximation at contact region \( j \). Likewise let \( \tau_j \) be the number of spanning forces, then \( f_j \) is defined by:

\[
f_j = \sum_{i=1}^{\tau_j} \lambda_{ji} f_{ji}
\]

(2)

where: \( \lambda_{ji} \) = the intensity multiplier for \( f_{ji} \).

In rigid body analysis, a wrench vector is often used to describe the effect of a resultant point load on a rigid body.

---

Nomenclature

\( a_i(A) \) = actuator intensity constraint function
\( A \) = actuator intensity constraint matrix
\( b_i \) = intensity limit constraint vector
\( b_i(A) \) = intensity limit constraint function
\( c(A) \) = objective function to the general problem
\( C \) = the number of clamps
\( d \) = a direction vector
\( d_k \) = the value of \( d \) at the end of iteration \( k \)
\( d_k^\Phi \) = the component of \( d_k \) associated with \( \Phi \)
\( d_k^{p} \) = the component of \( d_k \) associated with \( p \)
\( d_k^{\sigma} \) = the component of \( d_k \) associated with \( \sigma \)
\( d_i \) = displacement of point \( i \)
\( DP \) = the sum of the magnitudes of the point displacements
\( DP_{\text{max}} \) = the maximum \( DP \) recorded during clamping and machining
\( e_i(A) \) = static equilibrium constraint function
\( E \) = static equilibrium constraint matrix
\( f_j \) = the force exerted by contact region \( j \)
\( f_{ji} \) = the \( i \)th spanning force for \( f_j \)
\( g_i(x) \) = a function associated with an inequality constraint within Topkis and Venoitt’s algorithm
\( h_i(x) \) = a function associated with an equality constraint within Topkis and Venoitt’s algorithm
\( K \) = a matrix whose columns are the spanning wrenches associated with frictionless contact at the locators
\( L = \) the number of locators
\( n_i(A) \) = nonnegativity constraint function
\( p_j \) = the position of contact region \( j \)
\( p_{j^*} \) = an extreme point of the convex hull that bounds \( p_j \)
\( p \) = a vector consisting of all \( p_j \)
\( r_i(A) \) = contact region position constraint function
\( R \) = the rank of \( K \)
\( s_i(A) \) = sigma affinity constraint function
A wrench vector consists of a force and its moment about the reference frame origin. The wrench, \( \mathbf{w}_j \), exerted by contact region \( j \) is defined by:

\[
\mathbf{w}_j = \begin{bmatrix} \mathbf{f}_j \\ \mathbf{p}_j \times \mathbf{f}_j \end{bmatrix} = \sum_{l=1}^{\tau_j} \lambda_{jl} \begin{bmatrix} \mathbf{f}_{jl} \\ \mathbf{p}_{jl} \times \mathbf{f}_{jl} \end{bmatrix}
\]

(3)

where \( \lambda_{jl} \geq 0 \) for \( l = 1 \ldots \tau_j \)

Note that for each spanning force, there is an associated spanning wrench.

### 3.2 Actuator Intensity Constraints

A power clamp is a single degree of freedom mechanism which is either pneumatically or hydraulically actuated. The intensity of the actuated force or torque directly limits the intensity of the force that can be exerted at the contact region.

Assume that contact regions \( 1 \ldots C \) are associated with the C clamps, and let \( \alpha_j \) be the actuator intensity for clamp \( j \). Ordinarily the gravitational forces acting on the clamp links are negligible relative to the propagated machining forces, and thus can be ignored. Using the method of virtual work, the following relationship between \( \alpha_j \) and \( \lambda_{jl} \) can be derived:

\[
\alpha_j = \mathbf{Z}_j \cdot \sum_{l=1}^{\tau_j} \lambda_{jl} \mathbf{f}_{jl}
\]

(4)

\[
\lambda_{jl} \geq 0 \quad \text{for} \quad l = 1 \ldots \tau_j
\]

where: \( \mathbf{Z}_j = a_3 \times 1 \) actuation intensity constraint vector. In general \( \mathbf{Z}_j \) is dependent upon \( \mathbf{p}_j \) and the kinematic structure of the clamp.

If the clamp actuation intensity variables are combined into the vector, \( \mathbf{\alpha} \), such that \( \mathbf{\alpha} = [\alpha_1 \ldots \alpha_C]^T \), and the spanning force intensity multipliers are combined into the vector, \( \mathbf{\lambda} \), such that \( \mathbf{\lambda} = [\lambda_{11} \ldots \lambda_{1\tau_1} \ldots \lambda_{C+1,l1} \ldots \lambda_{(C+L),l\tau_j}]^T \), then the actuation intensity constraints for the C clamps can be written together as:

\[
\mathbf{\alpha} = \mathbf{A} \mathbf{\lambda}
\]

(5)

where: \( \mathbf{A} \) is a \( C \times \sum_{j=1}^{C+L} \tau_j \) constraint matrix.

### 3.3 Workpiece Static Equilibrium Constraints

During machining, the workpiece is subject to wrenches exerted by the contact regions, the cutting tool, and gravity. Without loss of generality, it will be assumed that the gravitational wrench acting on the workpiece is small compared to the other wrenches and thus can be ignored.

Let \( \mathbf{K} \) be the wrench exerted on the workpiece by the cutting tool during one of the machining operations. \( \mathbf{K} \) may be derived from one of the many tool force models which appear in the literature. Examples include Kline et al. (1982), Sutherland and De Vore (1986), and Yucesan et al. (1990).

From Newton's law, the sum of the wrenches exerted on the workpiece must equate to zero if the workpiece is to remain in static equilibrium. Consequently the following constraints must be satisfied for each operation throughout the range of \( \mathbf{K} \):

\[
\sum_{j=1}^{C+L} \mathbf{w}_j + \mathbf{K} = [\mathbf{0}]
\]

(6)

The constraints defined by (3) and (6) may be combined to obtain:

\[
\mathbf{E} \mathbf{\lambda} = -\mathbf{K}
\]

\[
\mathbf{A} \mathbf{\lambda} = [\mathbf{0}]
\]

(7)

where: \( \mathbf{E} \) is a \( 6 \times C \sum_{j=1}^{C+L} \tau_j \) constraint matrix.

For the workpiece to remain in static equilibrium, a solution must exist to the constraints defined by (5) and (7) throughout the range of \( \mathbf{K} \). In general \( \mathbf{K} \) is nonlinear and potentially discontinuous. Consequently it is difficult to use these constraints to guarantee workpiece immobilization during machining.

However this problem can be solved if a convex hull that bounds all values of \( \mathbf{K} \) is found (De Meter, 1994b). It has been proven that if a solution to (5) and (7) exists for each extreme point of this convex hull, then a solution must exist for all \( \mathbf{K} \). Let \( \mathbf{K}^1 \ldots \mathbf{K}^k \) be the extreme points of a convex hull which bounds \( \mathbf{K} \) for all operations. Consequently the workpiece is guaranteed to remain in static equilibrium if solutions exist to the following subproblems:

\[
\mathbf{E} \mathbf{\lambda} = -\mathbf{K}^k \quad \text{for} \quad k = 1 \ldots C+L
\]

\[
\mathbf{A} \mathbf{\lambda} = \mathbf{\alpha}
\]

\[
\mathbf{\lambda} \geq [\mathbf{0}]
\]

(8)

### 3.4 Min-Max Load

Given that a solution to any of the subproblems in (8) exists, it will not be unique. However it is assumed that when an external load is applied to a workpiece, the load is distributed as uniformly as possible among
the locator contact regions, keeping in mind that loading at the clamp contact regions is dictated primarily by the actuation intensities. It is also assumed that this distribution is the lowest possible in magnitude.

It has been shown that this solution is essentially equivalent to the one with the lowest uniform distribution of normal loads (De Meter, 1994b). This solution may be found through the application of linear programming. Let \( \phi \) be the minimum maximum normal load for a solution to a subproblem in (8). Then \( \phi \) must satisfy:

\[
\phi \geq \eta_j \sum_{i=1}^{C+L} \lambda_{ji} f_{ji} \quad \text{for} \quad j = C + 1 \ldots C + L
\]

In turn, [9] can be rewritten as:

\[
\phi \geq b_j A \quad \text{for} \quad j = C + 1 \ldots C + L
\]

where \( b_j \) is a \( \sum_{j=1}^{C+L} \) constraint vector.

It has been proven that if \( K \) is bounded by a convex hull, then \( \phi \) must take on its greatest value at one of the extreme points (De Meter, 1994b). From this point onward, solutions to the individual subproblems in (8) will be delineated by the superscript \( k \). The minimum maximum normal load, \( \phi_k \), for each subproblem can be found by solving the following linear program:

\[
\text{minimize} \quad \phi_k
\]

\[
\text{subject to:} \quad \phi_k - b_j A^k \geq 0 \quad \text{for} \quad j = C + 1 \ldots C + L
\]

\[
E A^k = -K^k
\]

\[
A A^k = \alpha
\]

\[
\lambda^k \geq [0]
\]

Let \( \Phi \) be the maximum \( \phi_k \) for the \( \gamma \) subproblems. Simulation experiments have shown that for a given set of machining operations and clamping intensities, the smaller the value of \( \Phi \), the smaller the workpiece displacement (De Meter, 1994b). Consequently \( \Phi \) can be used as an indirect means of predicting the impact of locator and clamp placement on workpiece displacement. Experiments have also shown that \( \Phi \) is a good indicator of the impact of clamping intensities on workpiece displacement. Consequently the min-max load criteria should provide a simple basis for design optimization.

3.5 Contact Region Position Constraints. The fixture designer is free to choose the positions of the contact regions on the designated workpiece surfaces. However only portions of these surfaces are usually suitable. This is due to clearance requirements between the fixture and the desired tool path. In addition, it is bad practice to position a fixture element close to a workpiece edge.

To reflect these position constraints, it is assumed that each \( p_j \) must lie within a convex region on a designated surface. This is illustrated in Fig. 3. It is assumed that this region has \( v_j \) extreme points. If \( p_1 \) \ldots \( p_{v_j} \) are the extreme points of the region, then the following convex set relationships must be satisfied:

\[
p_j = \sum_{m=1}^{v_j} \sigma_j^m p_j^m
\]

\[
\sum_{m=1}^{v_j} \sigma_j^m = 1
\]

\[
\sigma_j^m \geq 0 \quad \text{for} \quad j = 1 \ldots v_j
\]

where: \( \sigma_j^m \) is a multiplier of \( p_j^m \).

3.6 Total Restraint Requirement. A necessary attribute of a fixture is that it is capable of resisting any small wrench acting on the workpiece. This is important, since a workpiece is often subjected to smaller random forces from incidental contact and vibration prior to machining. If possessing this quality, the fixture is said to totally restrain the workpiece. Note that total restraint does not take into account the intensity of clamp actuation. It simply implies that the contact regions are capable of providing wrenches that can resist any arbitrary external wrench acting on the workpiece (Chou et al., 1989).

A fixture has this property if the spanning wrenches of its contact regions form a nonnegative linear hull which spans six dimensional space. This will always be true if a strictly positive vector exists within the null space of the matrix, \( E \). This may be proven through execution of the following linear program (De Meter, 1994b):

\[
\text{maximize} \quad T
\]

\[
\text{subject to:} \quad \lambda_j - T \geq 0 \quad \text{for} \quad j = 1 \ldots C + L, l = 1 \ldots \gamma
\]

\[
\sum_{j=1}^{C+L} \sum_{l=1}^{\gamma} \lambda_{ji} \left( p_j \times f_{ji} \right) = [0]
\]

\[
\lambda_{ji} \geq 0 \quad \text{for} \quad j = 1 \ldots C + L, l = 1 \ldots \gamma
\]

\[
T \leq 1
\]

This can take on one of two values. If it takes on the value 1, then the fixture provides total restraint. If it takes on the value 0, then it does not.

3.7 Kinematic Restraint Requirement. In addition to providing forces for restraint, the locators also serve to kinematically restrain the workpiece (Chou et al., 1989). This permits the quick establishment of the workpiece reference frame with respect to the machine tool reference frame. When evaluating kinematic restraint, only frictionless contact is considered. For the case of frictionless point contact, \( f_j \) is constrained to pass through \( p_j \) in the direction \( \eta_j \). Consequently, the spanning wrench \( b_j \) associated with frictionless point contact is \( \left[ f_j^T, \left( p_j \times f_j \right)^T \right]^T \).

It is known (Chou et al., 1989) that in order for a locator scheme to provide complete kinematic restraint, it must provide an independent spanning wrench system of \( K \) wrenches which are the spanning wrenches associated with frictionless contact at the L locators. Let \( R \) be the rank of \( K \). If the locators provide complete kinematic restraint, then \( R = 6 \). \( R \) is evaluated by reducing \( K \) to upper echelon form using row and column operations. \( R \) will equal the number of pivot columns within the reduced matrix.

4 Problem Solution

The designer's objective is to determine \( p_j \) and \( \alpha \) which result in minimum contact region loading and which satisfy the kinematic restraint and total restraint requirements. This problem is solved by first formulating and solving a simpler problem, the determination of optimal \( \alpha \) for a feasible fixture configuration. The constraints and solution procedure for this problem will be used in the formulation and solution of the general problem.
A feasible fixture configuration is one which satisfies the contact region position constraints and the kinematic and total restraint requirements. A designer can define a feasible solution by choosing \( \sigma_{ij}^m \) such that the nonnegativity and affinity constraints within (12) are satisfied; computing \( p_i \) and \( w_j \), and evaluating \( R \) and \( T \). If no feasible solution can be found, the designer should choose alternative surfaces for placement and/or increase the number of clamps. This technique can also be used to determine an initial feasible fixture configuration for the general problem.

4.1 Determination of Optimal Intensities. For a given fixture configuration, in which \( p_i \) are treated as constants, the matrices \( A \) and \( E \) are also constant. Consequently finding \( \alpha \) to minimize contact region loading is straightforward. However it must be recognized that since each subproblem in (11) is impacted by a change in \( \alpha \), they must be solved simultaneously. In addition nonnegativity constraints must be applied to \( \alpha \) since it is now treated as a variable.

Since \( \Phi \geq \Phi^k \forall k \), the constraint, \( \Phi - b_i \lambda^k \geq 0 \), can replace \( \Phi - b_i \lambda^k \geq 0 \) within (11). Consequently minimal \( \Phi \) can be found by solving the following linear program:

minimize \( \Phi \)

subject to:
\[
\begin{align*}
\Phi - b_i \lambda^k &\geq 0 & i = C + 1 \ldots C + L, \\
\alpha - A \lambda^k &= [0] & k = 1 \ldots \gamma \\
E \lambda^k &= -K^k & k = 1 \ldots \gamma \\
\lambda^k &\geq [0] & k = 1 \ldots \gamma \\
\alpha &\geq [0]
\end{align*}
\]

(14)

4.2 Determination of Optimal Configuration and Intensities. The general problem consists of the math programming problem defined by (14) in combination with the contact region position constraints defined by (12). However since \( p_i \) are variable, the moment equilibrium coefficients within the matrix \( E \) are variable. Consequently the moment equilibrium equations are quadratic as opposed to being linear. This necessitates the use of a nonlinear programming technique to solve this problem. However the near linear nature of the problem can be exploited as will be shown.

The explicit constraints of the kinematic restraint and total restraint requirements are difficult to integrate into this general problem. However \( R \) and \( T \) can be used to validate a solution against the kinematic and total restraint requirements. Evaluation of \( R \) and \( T \) will be treated as subproblems within the general solution algorithm.

4.2.1 Problem Characterization. To aid in the development of a solution procedure, the variables and constraints defined by (12) and (14) will be addressed using a different nomenclature. The variables \( \Phi \) and \( \alpha \) will retain their meaning. The variables \( \lambda, p, \sigma, A, \) and \( K \) will be defined as:

\[
\begin{align*}
\lambda &= \begin{bmatrix} \lambda^1 \\ \vdots \\ \lambda^\gamma \\ \lambda^L \end{bmatrix} \\
p &= \begin{bmatrix} p_1 \\ \vdots \\ p_{C+L} \end{bmatrix} \\
\sigma &= \begin{bmatrix} \sigma_1^1 \\ \vdots \\ \sigma_{C+L}^L \end{bmatrix} \\
\Phi &= \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_{C+L} \end{bmatrix} \\
\alpha &= \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{C+L} \end{bmatrix} \\
K &= \begin{bmatrix} K^1 \\ \vdots \\ K^{C+L} \end{bmatrix}
\end{align*}
\]

Let \( c(A) \) be the objective function of the problem such that \( c(A) = \Phi \). In addition define the functions on the left hand side of the constraints in (12) and (14) as:

\[
\begin{align*}
b_i(A) &= i \text{th intensity limit constraint} \\
a_i(A) &= i \text{th actuator intensity constraint} \\
e_i(A) &= i \text{th static equilibrium constraint} \\
r_i(A) &= i \text{th contact region position constraint} \\
s_i(A) &= i \text{th sigma affinity constraint} \\
n_i(A) &= i \text{th nonnegativity constraint}
\end{align*}
\]

Consequently the general problem can be rewritten as:

minimize \( c(A) \)

subject to:
\[
\begin{align*}
b_i(A) &\geq 0 & \text{for } i = 1 \ldots Ly \\
a_i(A) &\geq 0 & \text{for } i = 1 \ldots Cy \\
e_i(A) &= -K[i] & i = 1 \ldots 6y \\
r_i(A) &= 0 & i = 1 \ldots 3(C + L) \\
s_i(A) &= 1 & i = 1 \ldots (C + L) \\
n_i(A) &\geq 0 & i = 1 \ldots C + \gamma \sum_{k=1}^{C+L} \tau_k \\
\sum_{k=1}^{C+L} \tau_k + 1 &< 0
\end{align*}
\]

(15)

There are many techniques which can be applied to solve this problem. However due its near linearity, a feasible direction technique is well suited. A feasible direction technique is one which searches the feasible solution region for a \( A \) which satisfies the Fritz-Notes necessary local optimality condition (Bazaraa et al., 1993).

This condition is the following. Let \( A \) be a feasible solution to the general problem. Define \( d \) to be a direction in the solution space. If \( A \) is a local optimum solution, there should be no \( d \) which satisfies the following:

\[
\forall c(A) : d < 0
\]

subject to:
\[
\begin{align*}
v_{b_i}(A) \cdot d &> 0 & \text{for all } i \text{ such that } b_i(A) = 0 \\
v_{a_i}(A) \cdot d &> 0 & \text{for } i = 1 \ldots Cy \\
v_{e_i}(A) \cdot d &> 0 & \text{for } i = 1 \ldots 6y \\
v_{r_i}(A) \cdot d &> 0 & \text{for } i = 1 \ldots 3(C + L) \\
v_{s_i}(A) \cdot d &> 0 & \text{for } i = 1 \ldots C + L \\
v_{n_i}(A) \cdot d &> 0 & \text{for all } i \text{ such that } n_i(A) = 0
\end{align*}
\]

(16)

Note that if \( A \) satisfies the Fritz-Notes condition, there is no guarantee that it is either a global or local optimum solution. This is due to the fact that there is no guarantee that the moment equilibrium equations will be quasiconcave at \( A \).

A number of different nonlinear programming techniques have been developed to handle problems with combinations of equality and inequality constraints. The algorithm which was developed to solve this problem is a derivative of the one developed by Topkis and Veinott (1967). Their algorithm has three desirable properties. First it is guaranteed to converge to a Fritz-Notes point. Second it can be modified to exploit the near linearity of the general problem. Third it can be modified to integrate the evaluation of \( R \) and \( T \) during the search for a new feasible solution. This algorithm is discussed next.

4.2.2 Topkis and Veinott's Method of Feasible Directions. Topkis and Veinott's algorithm was developed to solve the following problem:

minimize \( c(x) \)

subject to:
\[
\begin{align*}
g_i(x) &\leq 0 & \text{for } i = 1 \ldots M \\
h_i(x) &= 0 & \text{for } i = 1 \ldots N
\end{align*}
\]

(17)

where: \( x \in R^n \).

It is a three step iterative procedure designed to start with an
initial feasible solution and converge to a Fritz-John point. Iteration \(k\) starts with the feasible solution \(x_k\). The first step in the search sequence is to find a search direction \(d\) which leads to the greatest infinitesimal decrease in the objective function. This is done through the evaluation of the following linear program:

minimize \(z\)
subject to:
\[
\nabla c(x_k) \cdot d - z \leq 0
\]
\[

abla g_i(x_k) \cdot d - z \leq -g_i(x_k) \quad \text{for } i = 1 \ldots M
\]
\[

abla h_i(x_k) \cdot d = 0 \quad \text{for } i = 1 \ldots N
\]
\[
-1 \leq d[i] \leq 1 \quad \text{for } i = 1 \ldots n
\] (18)

If the solution to [18] is \(z = 0\), then \(x_k\) is a Fritz-John point and the procedure stops. If not, \(d_k = d\) and the second step is performed.

In step 2, a line search is performed along the direction \(d_k\) in order to find an intermediate solution \(x_{k+1}\) which provides a smaller objective value than \(x_k\). This is done by finding the solution, \(\delta\), to the following line search problem:

minimize \(c(x_k + \delta d_k)\)
subject to:
\[
0 \leq \delta \leq \delta_{\text{max}}
\] (19)

where:
\[
\delta_{\text{max}} = \inf \{ \delta : g_i(x_k + \delta d_k) \leq 0 \quad \text{for } i = 1 \ldots M \}
\]

This problem is solved using any unconstrained line search technique. Upon determination of \(\delta\), \(x_{k+1} = x_k + \delta d_k\) is computed using the following relationship:

\[
x_{k+1} = x_k + \delta d_k
\] (20)

If the equality constraints are linear, then \(x_{k+1}\) will exist within the feasible region and \(x_{k+1} = x_{k+1}\). In this case the procedure returns to step 1. However if any equality constraint is nonlinear, \(x_{k+1}\) may not be feasible. In which case step 3 is performed.

In step 3, \(x_{k+1}\) is determined by hopping back into the feasible region from \(x_{k+1}\). This may be done by any means. Once \(x_{k+1}\) is determined, the procedure returns to step 1.

4.2.3 Solution Algorithm. The solution algorithm uses the three iterative steps of Topkis and Veinott’s algorithm. However these steps are adapted to exploit the near linearity of the general problem as well as insure that a feasible solution satisfies the kinematic and total restraint requirements. Like Topkis and Veinott’s algorithm, this procedure starts with an initial feasible solution.

An initial feasible solution is obtained by first using the procedure described previously to obtain a feasible fixture configuration and thus values for \(p\) and \(\sigma\). Subsequently \(\alpha\) can be selected and (11) solved to find \(\phi\) and \(\lambda\). Since the layout will be feasible, a feasible \(\alpha\) is guaranteed to exist if the actuation intensity constraints are properly formulated. Alternatively \(\phi\), \(\sigma\), and \(\lambda\) can be determined through the solution of (14). In either case, the procedure proceeds as follows.

4.2.3.1 Step 1: Determination of a Search Direction. At the beginning of iteration \(k\), a feasible solution \(\lambda_k\) is in hand. To find a search direction, the following linear program is evaluated:

minimize \(z\)
subject to:
\[
z - \nabla c(\lambda_k) \cdot d \geq 0
\]
\[
z - \nabla b_i(\lambda_k) \cdot d \geq -b_i(\lambda_k) \quad \text{for } i = 1 \ldots L
\]
\[
\nabla a_i(\lambda_k) \cdot d = 0 \quad \text{for } i = 1 \ldots C
\]
\[
\nabla r_i(\lambda_k) \cdot d = 0 \quad \text{for } i = 1 \ldots 3(C + L)
\]

If \(z = 0\), then stop. \(\lambda_k\) is a Fritz-John point. If not, \(d_k = d\), go to step 2. Note that the designer may wish to stop the procedure if \(z\) is close to zero. In this case an \(\epsilon\) may be chosen such that the procedure is stopped for \(z \geq -\epsilon\).

4.2.3.2 Step 2: Line Search. To find \(\lambda'_{k+1}\), we must first find the solution, \(\delta\), to the following problem:

minimize \(c(\lambda_k + \delta d_k)\)
subject to:
\[
0 \leq \delta \leq \delta_{\text{max}}
\] (22)

Let \(d^\phi_k\) be the component of \(d_k\) associated with \(\phi\). Then the objective function may be rewritten as:
\[
c(\lambda_k + \delta d_k) = c(\lambda_k) + \delta d^\phi_k
\] (23)

By necessity \(d^\phi_k\) must be negative. Hence the solution to (23) is \(\delta = \delta_{\text{max}}\). By definition:
\[
\delta_{\text{max}} = \sup \{ \delta : \delta b_i(\lambda_k + \delta d_k) \geq 0, \forall i, \delta n_i(\lambda_k + \delta d_k) \geq 0, \forall i \}
\] (24)

However since \(b_i(\lambda)\) and \(n_i(\lambda)\) are linear the following is true:
\[
b_i(\lambda_k + \delta d_k) = b_i(\lambda_k) + \delta b_i(\lambda_k)
\]
\[
n_i(\lambda_k + \delta d_k) = n_i(\lambda_k) + \delta n_i(\lambda_k)
\] (25)

As a result, \(\delta_{\text{max}}\) is the largest \(\delta\) which satisfies:
\[
\delta \geq -\frac{-b_i(\lambda_k)}{b_i(\lambda_k)} \quad \text{for } i = 1 \ldots L
\]
\[
\delta \geq -\frac{-n_i(\lambda_k)}{n_i(\lambda_k)} \quad \text{for } i = 1 \ldots C + \sum_{j=1}^{C+C} \gamma_j + \sum_{j=1}^{C+L} \nu_j + 1
\] (26)

Next we must find \(\lambda'_{k+1}\). If Topkis and Veinott’s strategy is employed, then \(\delta = \delta_{\text{max}}\) and \(\lambda'_{k+1} = \lambda_k + \delta d_k\). However due to the quadratic moment equilibrium constraints, \(\lambda'_{k+1}\) will probably be infeasible. However since \(r_i(\lambda)\) and \(s(\lambda)\) are linear, both \(p_k\) and \(\sigma_k\) will be feasible. This fact can be exploited to insure that the new fixture configuration meets the kinematic and total restraint requirements and to facilitate the computation of \(\lambda'_{k+1}\).

To do so, let \(d_k^p\) and \(d_k^\sigma\) be the components of \(d_k\) associated with \(p\) and \(\sigma\). Now replace Topkis and Veinott’s line search with the following:

maximize \(\delta\)
subject to:
\[
p_{k+1} = p_k + \delta d_k^p
\]
\[
\sigma_{k+1} = \sigma_k + \delta d_k^\sigma
\]
\[
R = R_0
\]
\[
T = T_0
\]
\[
0 \leq \delta \leq \delta_{\text{max}}
\] (27)

As \(\delta\) varies, \(p_{k+1}\) is computed along the spanning wrenches of the new contact regions and \(R\) and \(T\) are evaluated. Experience has indicated that the solution to (27) is \(\delta = \delta_{\text{max}}\). However to date there is no proof that this will always be the case. Nevertheless this is the best initial solution to check. If it proves to be infeasible, then the strategy should be to
increment \( \delta \) by small amounts until a fixture configuration is found which satisfies the kinematic and total restraint requirements.

At this stage \( \Phi_{k+1} \) may still be infeasible. However by performing this line search, we have determined the set of contact region positions which result in the greatest decrease in \( \Phi \). At this point we go to step 3.

4.2.3.3 Step 3: Determination of a New Feasible Solution. To find \( \Phi_{k+1} \), the linearity of (14) for fixed \( p \) is exploited. To determine \( \Phi_{k+1}, \alpha_{k+1}, \) and \( \Phi_{k+1}, p_{k+1} \) is substituted into (14) which is solved as a linear program. Once this has been done, the procedure returns to step 1. Note that while this procedure is guaranteed convergence, an iteration limit may be imposed to stop the procedure at this point.

5 Algorithm Validation

To determine its effectiveness, the algorithm was used to improve a set of fixture designs for various milling operations. Each design was initially analyzed using a workpiece displacement model similar to the one developed by Lee and Haynes (1987). Each design was subsequently used to generate a starting solution for the algorithm. After the algorithm was executed, the new design parameters were recorded. These parameters were then analyzed using the workpiece displacement model to determine the improvement in performance.

The experimentation was carried out as follows. The workpiece was considered to be a 15.24 cm \( \times \) 15.24 cm \( \times \) 3.81 cm block made of 7075-T6 aluminum. The workpiece was subject to one of three linear conventional milling passes as shown in Fig. 4. Each pass was 12.74 cm in length. The tool and machining parameters for each pass are given below:

Machining Parameters: Spindle Speed = 530 rpm
Axial Depth of Cut = .238 cm
Radial Depth of Cut = .762 cm
Feed Rate = 23.5 cm/min

Tool: 4 Flute End Mill
Diameter = 1.905 cm
Length = 4.445 cm
Helix Angle = 30 degrees

Cutting Fluid: None

It was assumed that the fixture consisted of six spherical-tipped locators and two spherical-tipped, axial thrust clamps. The elements were arranged in one of three different layouts. These layouts were similar to those used in actual production for similar cuts. Each layout used a 3-2-1 locating scheme, with three locators placed on the primary datum feature (datum zero plane), two locators placed on the secondary datum feature (datum XZ datum plane), and one locator placed on the tertiary datum feature (datum: YZ plane).

The clamps were placed on the planar surfaces opposing the secondary and tertiary datum features. The location data for these layouts are provided in Table 1. The coordinates are defined relative to the coordinate system illustrated in Fig. 4. The intensity of each clamp was set to 1041 N. The coefficient of friction at all contact regions was assumed to be .2.

The model developed by Yucesan et al. (1990) was used to compute values of \( K \) for each cut. These values were used with the displacement model to compute workpiece displacement for each fixture design. During analysis, the displacements \( d_1, d_2, \) and \( d_3 \) of the points \( p_1, p_2, \) and \( p_3 \) on the workpiece were tracked as shown in Fig. 5. Prior to displacement, each of these points lay on a principal axis of the fixture reference frame, 2.54 cm from the origin.

For each data set recorded, the sum of the magnitudes (DP) of \( d_1, d_2, \) and \( d_3 \) was also computed. In addition, the maximum \( Dp \) for each operation was also determined. This value, which is defined as \( Dp_{max} \), served as a measure of displacement severity.

The values of \( K \) were also used to compute a 21 point convex hull (De Meters, 1994b) to bound \( K \) for each cut. The extreme points of this hull were used with (11) to compute \( \Phi \) for each fixture design/cut combination. The fixture designs and their corresponding values of \( \Phi \) were subsequently used as starting solutions for the optimization algorithm.

The optimization algorithm was executed to modify the layouts and clamp actuation intensities in order to obtain improved values of \( \Phi \). Note that the locators and clamps were constrained to lie within rectangular regions on their respective surfaces. The boundaries of these regions were taken from the workpiece edges as shown in Fig. 3. Once the new layouts and actuation intensities were obtained, they were then evaluated using the displacement model.

6 Results

The results of the experimentations are summarized in Table 2. The first two columns define the initial layouts and the cuts. The next two columns illustrate the values of \( \Phi \) and \( Dp_{max} \) for the initial designs. The next two columns illustrate the values of \( \Phi \) and \( Dp_{max} \) for the new designs computed by the optimization algorithm. The last column defines the number of iterations taken by the algorithm to converge to a solution.

The new layouts computed by the algorithm are shown in Tables 3–5, while the new actuation intensities are shown in Table 6. As can be seen from this data, the clamp actuation...
intensities were subject to greater change than the fixture layouts. Consequently it is assumed that the initial intensities were much higher than the minimum necessary to keep the workpiece in static equilibrium.

It should be noted that these solutions may not be local optimum solutions. Nevertheless the use of the algorithm led to significant improvements in $\Phi$ and $DP_{\text{max}}$ in a relatively small number of iterations.

7 Conclusions

The following conclusions can be drawn from this paper. The algorithm presented is a simple but effective means for determining good fixture designs. It is easy to formulate and easy to solve. However, because it is based on an indirect measure of workpiece displacement, it cannot be relied upon to find an optimal solution.

Nevertheless this algorithm is an important tool. Because it is inexpensive to apply, it can be used to quickly generate good feasible designs. These in turn can be used in more complex algorithms, thus reducing the overall cost of design optimization. Future research will be directed toward modifying the algorithm to handle nonprismatic workpieces.

References

De Meter, E. C., 1994a, "Restraint Analysis of Fixtures which Rely on Surface Contact," ASME JOURNAL OF ENGINEERING FOR INDUSTRY, in press.
De Meter, E. C., 1994b, "The Min-Max Load Criteria as a Measure of Machining Fixture Performance," ASME JOURNAL OF ENGINEERING FOR INDUSTRY, in press.