A NOVEL NUMERICAL APPROACH OF COMPUTING AMERICAN OPTION*

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ABSTRACT

It is well acknowledged that the European options can be valued by an analytic formula, but situation is quite different for the American options. Mathematically, the Black-Scholes model for the American option pricing is a free boundary problem of partial differential equation. This model is a non-linear problem; it has no closed form solution. Although approximate solutions may be obtained by some numerical methods, but the precision and stability are hard to control since they are largely affected by the singularity at the exercise boundary near expiration date. In this paper, we propose a new numerical method, namely SDA, to solve the pricing problem of the American options. Our new method combines the advantages of the Semi-analytical Method and the Sliced-fixed Boundary Finite Difference Method while overcomes demerits of the two. Using the SDA method, we can resolve the problems resulted from the singularity near the optimal exercise boundary. Numerical experiments show that the SDA method is more accurate and more stable than other numerical methods. In this paper, we focus on the American put options, but the proposed method is also applicable to other types of options.

Keywords: American Option, Black-Scholes Equation, Free Boundary, Analytical Method of Lines, Finite Difference Method

1. Introduction

Option is a kind of financial derivatives, which can be viewed as a long-term agreement between investors and financial institutions who offer the options. In

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money market, European and American options are usually cared. The European options get executed on the expiration date, while American options can be executed on any normal deal date on or before the expiration date. On the day of maturity, the holders have the right to actualize the agreement with some underlying asset in the strike price, say $X$, following the agreement. We assume that $T$ is the expiration date of the agreement and $S_T$ is the underlying asset price at the time of maturity. The holder of a European put option can execute the agreement to get the income $X - S_T (> 0)$ when $S_T < X$; on the other hand, when $S_T > X$, the holder does not need to execute the agreement. With the European call option, holders can choose to execute the agreement only when $S_T > X$ to get the income of $S_T - X (> 0)$. In other words, the holder who purchases such option have the right to decide whether to execute the agreement or not, though it does cost the holder to purchase the agreement in the first place. How to price the option rises as a problem, and the Black-Scholes theory of option pricing[2] can solve it. In 1973, under some assumptions, such as no-arbitrage and no transaction costs, Black, Scholes and Merton made use of the ITO Lemma to get a partial differential equation which describes the price of the European options $P(S, t)$:

$$
\frac{\partial P}{\partial t} + (r - q)S \frac{\partial P}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - rP = 0, \quad (S, t) \in (0, \infty) \times (T, 0).
$$

(1)

Here, $t \in [0, T]$ is a variable of time; $S = S_t$ is the underlying asset price which is a random variable subjected to logarithmic normal distribution; $r > 0$ is riskless rate; $\sigma > 0$ is volatility; $q \geq 0$ is the dividend yield. Equation (1) is called Black-Scholes equation, which is a backward partial differential equation. The solution of equation (1) can be expressed in a closed form after the terminal-boundary conditions are specified[14].

Different from the European options, investors holding the American options can execute agreement every trading day on or before expiration date. Assuming we consider the American put options, let

$$
\bar{S}(t) = \sup\{ S \geq 0 \mid P(S, t) = (X - S)^+ \}, \quad t < T.
$$

(2)

$\bar{S}(t) \in C^\infty$ and $[0, T]$ is a no-descending function as the unknown exercise boundary. It is clear, $\lim_{t \to T} \bar{S}(t) = X$. Here, $(X - S)^+ = \max(0, X - S)$. Let

$$
t^* = \inf\{ t \in [0, T] \mid S = \bar{S}(t) \}, \quad t < T.
$$

We can prove that[13] if and only if executing agreement at the time of $t = t^*$, investors can get the best income. Let domain

$$
D = \{(S, t) \mid S > \bar{S}(t), t \in (0, T)\},
$$

it is optimal to execute the agreement when $(S, t)$ penetrates outwards the boundary of $D$ for the first time. $\bar{S}(t)$ is called critical price. $D$ is called continuation region. In the domain $D$, the price of the American option $P(S, t)$ must satisfy the equation
and on the boundary $S = \mathcal{S}(t)$ and $S = \infty$, it must satisfy the boundary condition:

$$P(\mathcal{S}(t), t) = X - \mathcal{S}(t), \quad \frac{\partial P(\mathcal{S}(t), t)}{\partial S} = -1, \quad P(\infty, t) = 0.$$  \hfill (3)

In addition, the American put options must satisfy the terminal condition:

$$P(S, T) = (X - S)^+.$$  \hfill (4)

It is clear that the solution to equation (1)-(4) is a free boundary problem. The boundary $\mathcal{S}(t)$ is unknown implies the non-linear nature of this problem. As a result, the solution has no closed form. Our work not only includes the solution to equation $P(S, t)$, but also the search for the position of a free boundary $\mathcal{S}(t)$.

The existing methods can be classified into three types: analytical approximation methods[1, 5, 9, 12, 15], semi-analytical approximation methods[4, 6, 10, 12, 13, 16] and purely numerical methods[3, 17, 18, 19]. The analytical approximations are very fast but offer no means of improving accuracy. For example, the long-term American options are more sensitive to volatility than the short-term options. The disability to increase accuracy has lead users to abandon the analytical approximation methods when dealing with long-term options. In response to these shortcomings, the semi-analytical approximation methods[6, 7] have been developed. For instance, the partial differential equation (1) are discretized along time (to maturity) in the analytical method of line, so the equations (1)-(4) become linear differential equations. These methods have a relatively high accuracy and stability, but the formula is very complex and rarely used. The finite difference method and the binomial tree method are most widely used, but they have three disadvantages as discussed below:

First, the first order derivative with respect to the variable $S$ of function $(X-S)^+$ is not continuous. This may greatly influence the precision of the difference method.

Second, the free boundary $\mathcal{S}(t)$ for the American options has the following asymptotic behavior (Barleo, Romano & Samscen, working paper, 1995),

$$X - \mathcal{S}(t) \sim X\sigma\sqrt{(T-t)} - \log(1/(T-t)).$$  \hfill (5)

(4) and (5) show that $\mathcal{S}'(t)$, $\bar{\theta} \equiv \partial P(S, t)/\partial t$ and $\Gamma \equiv \partial^2 P(S, t)/\partial S^2$ have the singularity when $t \to T$ and $S \to X$. It can make great influence of the accuracy and stability of the difference method.

Third, the domain $D = (\mathcal{S}(t), \infty) \times (0, T)$ is unbounded, it also creates errors because the difference method usually has to be cut off with a bounded sub-domain $D' \subset D$.

Hence, we introduce a new numerical method to solve (1)-(4). The main steps are described as the following: Firstly, letting $\tau = T - t$, with the technique of fixed boundary, we transform the free boundary problem (1)-(4) into the initial-boundary problem of a forward parabola equation in $(0, 1) \times (0, T)$. Secondly, the differential equation of fixed boundary is discretized along time with the difference method. In order to overcome the singularity at the neighbor $\tau = 0$, we apply analytical
method of line to the first two time intervals: \( \tau_1 = \Delta \tau \) and \( \tau_2 = 2\Delta \tau \), and then we still use normal difference method to solve the problem at the other time intervals. We refer to this method as the *Sliced-fixing Boundary-finite Difference Method & Analytical Method of Line*, SDA in short.

A large amount of calculation shows that the method illustrated in this paper is more accurate, and more stable. With it, we can get a smoother and more actual curve of exercise boundary of the American options. In a sense of concision, only the American put options is discussed in details below, but we believe the method is also useful for the valuation of other types of options.

2. Transformation From The Free Boundary Problem into a Fixed Boundary problem

Through the following four steps, the free boundary problem (1)-(4) is transformed into a fixed boundary problem. Without losing generality, we discuss this process under \( q = 0 \) (that is no dividends).

**Step1:** Let \( \tau = T - t \), and \( \tilde{S} = S/X \), \( \tilde{P}(S, \tau) = P(S, t)/X \), \( \tilde{S}(\tau) = S(t)/X \), \( X \) is eliminated. From now on, we use \( P \) to denote \( \tilde{P} \), \( \bar{S} \) to denote \( \tilde{S} \), and \( S \) to denote \( \bar{S} \). Equation (1)-(4) can be rewritten as:

\[
\begin{align*}
  P(S, \tau) &= 1 - S, \quad (S, \tau) \in (0, \bar{S}(\tau)) \times (0, T), \\
  P_{\tau} &= \frac{1}{2} S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP, \quad (S, \tau) \in (\bar{S}(\tau), \infty) \times (0, T), \\
  P(S, 0) &= (1 - S)^+, \quad S \in (0, \infty), \\
  P(\bar{S}(\tau), \tau) &= 1 - \bar{S}(\tau), P_S(\bar{S}(\tau), \tau) = -1, P(\infty, \tau) = 0.
\end{align*}
\]

(6)

**Step2:** Let \( \Delta \tau = \frac{T}{N+1} \), \( n = 0, 1, \cdots, N \), and use \( \bar{S}^n \) to denote \( \bar{S}(n\Delta \tau) \), \( P^n(S) \) to denote \( P(S, n\Delta \tau) \),

\[
\begin{align*}
  \frac{P^n(S) - P^{n-1}(S)}{\Delta \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 P^n(S)}{\partial S^2} - rS \frac{\partial P^n(S)}{\partial S} + rP^n(S) &= 0, S \in (\bar{S}^n, \infty), \\
  P^0(S) &= (1 - S)^+, \quad S \in (0, \infty), \\
  P^n(\bar{S}^n) &= 1 - \bar{S}^n, \quad P^n_S(\bar{S}^n) = -1, \quad n = 1, 2, \cdots, N, \\
  P^n(\infty) &= 0.
\end{align*}
\]

(7)

**Step3:** Let \( x = \ln(S/\bar{S}^n) \) or \( S = \bar{S}^n e^x \), \( n = 1, 2, \cdots, N \), so we have

\[
\begin{align*}
  S \frac{\partial P^n}{\partial S} &= \frac{\partial P^n}{\partial x}, \\
  S^2 \frac{\partial^2 P^n}{\partial S^2} &= \frac{\partial^2 P^n}{\partial x^2} - \frac{\partial P^n}{\partial x},
\end{align*}
\]

(8)

here,

\[
P^{n-1}(x) = \begin{cases} 
  1 - \bar{S}^{n-1}, & 0 \leq x \leq \ln(\bar{S}^{n-1}/\bar{S}^n), \\
  P^{n-1}_*(x), & x > \ln(\bar{S}^{n-1}/\bar{S}^n).
\end{cases}
\]

where \( P^{n-1}_*(x) \) is the value of the \((n-1)th\) phase. Putting (7) and (8) into (6), we get
\[
\begin{align*}
(1 + r\Delta\tau)P^n - \frac{\sigma^2}{2} \Delta\tau \frac{\partial^2 P^n}{\partial x^2} - \left( r - \frac{\sigma^2}{2} \right) \Delta\tau \frac{\partial P^n}{\partial x} &= P^{n-1}, \quad x \in (0, \infty), \\
P^0(x) &= 0, \quad x \in (0, \infty), \\
P^n(0) &= 1 - S^n, \quad \frac{\partial P^n(0)}{\partial x} = -S^n, \quad n = 1, 2, \cdots, N, \\
P^n(\infty) &= 0.
\end{align*}
\]

we have
\[
\begin{align*}
S^n &= 1 - P^n(0), \\
P^n(0) - \frac{\partial P^n(0)}{\partial x} &= 1.
\end{align*}
\]

Obviously, the free boundary \(S(\tau)\) becomes the fixed boundary \(x = 0\) under the transformation. Note that the other boundary is still infinite, i.e. \(x = \infty\). It is a trouble for numerical method to solve this boundary problem. Thus, at last we introduce the following transformation.

**Step4:** Let
\[
y = x/(1+x) \quad \text{or} \quad x = 1/(1-y),
\]

we have
\[
\begin{align*}
\frac{\partial P^n}{\partial x} &= (1-y)^2 \frac{\partial P^n}{\partial y}, \\
\frac{\partial^2 P^n}{\partial x^2} &= (1-y)^4 \frac{\partial^2 P^n}{\partial y^2} - 2(1-y)^3 \frac{\partial P^n}{\partial y}.
\end{align*}
\]

Putting transformation (10) and (11) into (9), we get
\[
\begin{align*}
(1 + r\Delta\tau)P^n - \frac{\sigma^2}{2} \Delta\tau (1-y)^4 \frac{\partial^2 P^n}{\partial y^2} - \\
\Delta\tau \left[ \left( r - \frac{\sigma^2}{2} \right) (1-y)^2 - \sigma^2 (1-y)^3 \right] \frac{\partial P^n}{\partial y} &= P^n, \quad y \in (0,1), \\
P^0(y) &= 0, \quad y \in (0,1), \\
P^n(0) - \frac{\partial P^n}{\partial y} &= 1, \quad n = 1, 2, \cdots, N, \\
P^n(1) &= 0, \quad S^n = 1 - P^n(0), \quad n = 1, 2, \cdots, N.
\end{align*}
\]

In the same way, we revise \(P^{n-1}(y)\) according to fellow:
\[
P^{n-1}(y) = \begin{cases} 
1 - S^{n-1}, & 0 \leq y \leq \ln \left( S^{n-1}/S^n \right) \left/ \left[ 1 + \ln \left( S^{n-1}/S^n \right) \right] \right. \\
P^*_{n-1}(y), & y > \ln \left( S^{n-1}/S^n \right) \left/ \left[ 1 + \ln \left( S^{n-1}/S^n \right) \right] \right.
\end{cases}
\]

Where \(P^*_{n-1}(y)\) is the computation results of the \((n-1)th\) time phase \(\tau_{n-1} = (n-1)\Delta\tau\).
Through these four steps, the free boundary problem has been transformed into a fixed boundary problem in the bounded domain \((0,1) \times (0,T)\).

3. Finite Difference Method

Let grid step \(h = \Delta y\) and \(k = \Delta \tau\), within the two dimension \(y-\tau\) region \((0,1) \times (0,T)\). Because \(h = 1/(M+1)\), \(k = T/(N+1)\), totally there are \(N+2\) time intervals: \(0,\Delta \tau, 2\Delta \tau, \cdots, \Delta \tau(N+1) = T\), and \(M+2\) kinds of stock prices: \(0,\Delta \tau, 2\Delta \tau, \cdots, \Delta \tau(M+1) = 1\), which make up of a \((M+2) \times (N+2)\) grid. \(I_i^n\) denotes the option value of time \(n\Delta \tau\) and stock price \(i\Delta \tau\), here \(i = 1, 2, \cdots, M+1, n = 1, 2, \cdots, N+1\). Define \(D_+ = (E_+ - I)/h, D_- = (I - E_-)/h\), \(D_0 = (E_+ - E_-)/2h\); Here, \(E_+\) and \(E_-\) is defined as below:

\[
E_+ P_i^n = P_{i+1}^n, \quad E_- P_i^n = P_{i-1}^n, \quad I = E_+ E_-.
\]

Calculate (12) with two level of difference method:

\[
\left\{\begin{array}{l}
(1 + rk) - \frac{\sigma^2}{2} k (1 - i h)^4 D_+ D_- \\
\left[\left(r - \frac{\sigma^2}{2}\right)(1 - i h)^2 - \sigma^2 (1 - i h)^3\right] k D_0
\end{array}\right\} P_i^n = P_i^{n-1},
\]

\[
i = 1, 2, \cdots, M; n = 1, 2, \cdots, N,
\]

\[
P_i^0 = 0, \quad i = 1, 2, \cdots, M,
\]

\[
P_0^n - D_0 P_0^n = 1, \quad n = 0, 1, \cdots, N,
\]

\[
P_{M+1}^n = 0, \quad n = 1, 2, \cdots, N+1,
\]

\[
S^n = 1 - P_0^n, \quad i = 1, 2, \cdots, N+1.
\]

Then rewrite (14) as below:

\[
\left\{\begin{array}{l}
(1 + rk) - \frac{\sigma^2}{2} k (1 - i h)^4 D_+ D_- \\
\left[\left(r - \frac{\sigma^2}{2}\right)(1 - i h)^2 - \sigma^2 (1 - i h)^3\right] k D_0
\end{array}\right\} P_i^n = P_i^{n-1},
\]

\[
i = 1, 2, \cdots, M; n = 1, 2, \cdots, N,
\]

\[
P_i^0 = 0, \quad i = 0, 1, \cdots, M,
\]

\[
P_0^n = 1 - (1 - P_0^n) / \left[\exp\left(-\frac{h}{1+h}\right) + 2h\right], \quad n = 1, 2, \cdots, N,
\]

\[
P_{M+1}^n = 0, \quad n = 1, 2, \cdots, N.
\]

Note that (15) is a linear system. Calculate it and we get:

\[
-(a_i + b_i) P_i^{n+1} + (c + 2a_i) P_i^n - (a_i - b_i) P_{i-1}^n = P_{i-1}^{n-1},
\]

\[
i = 1, 2, \cdots, M, n = 1, 2, \cdots, N,
\]

\[
P_i^0 = 0, \quad i = 0, 1, \cdots, M,
\]

\[
p_0^n = 1 - d(1 - P_1^n), \quad n = 1, 2, \cdots, N,
\]

\[
p_{M+1}^n = 0, \quad n = 1, 2, \cdots, N.
\]
Here,

\[
\begin{align*}
    a_i &= \frac{\sigma^2}{2}k(1 - ih)^4, \quad b_i = \left[ (r - \frac{\sigma^2}{2}) (1 - ih)^2 - \sigma^2(1 - ih)^3 \right] \frac{k}{2h}, \\
    c &= 1 + rk, \quad d = 2h + \exp \left( \frac{-h}{1 + h} \right),
\end{align*}
\]

with matrix form denotation (16)

\[
AP^n = P^{n-1} + e,
\]

here,

\[
P^n = (P^n_1, P^n_2, \ldots, P^n_M)^T, \quad P^{n-1} = (P^{n-1}_1, P^{n-1}_2, \ldots, P^{n-1}_M)^T,
\]

\[
e = \left[ ((a_1 - b_1)(d - 1))/d, 0, \ldots, 0 \right]^T,
\]

\[
a_{11} = [(c + 2a_1) - (a_1 - b_1)]/d, \quad a_{ii} = c + 2a_i, \quad i = 2, 3, \ldots, M,
\]

\[
a_{i,i+1} = -(a_i + b_i), \quad a_{i+1,i} = -(a_i - b_i), i = 2, 3, \ldots, M, \text{other, } a_{ij} = 0.
\]

\[
P^{n-1} = \begin{cases} 
1 - S^{n-1}, & 0 \leq ih \leq \ln \left( \frac{S^{n-1}/S^n}{1 + \ln \left( \frac{S^{n-1}/S^n}{1} \right)} \right) \\
S^{n-1} - P^0_n, & ih > \ln \left( \frac{S^{n-1}/S^n}{1 + \ln \left( \frac{S^{n-1}/S^n}{1} \right)} \right) 
\end{cases}
\]

\[
S^n = 1 - P^n_0 = (1 - P^n_1)/\left[ \exp \left( \frac{-h}{1 + h} \right) + 2h \right] \\
\approx (1 - P^{n-1}_1)/\left[ \exp \left( \frac{-h}{1 + h} \right) + 2h \right].
\]

Now, at each time interval: \( \Delta \tau, 2\Delta \tau, \ldots, (N + 1)\Delta \tau = T \), with GMRES(m) method, we above are not very accurate. In response to this shortcoming, we can apply the analytical method of line to the first two time intervals \( n = 1 \) and \( n = 2 \) [15], meanwhile at the other time intervals \( n \geq 3 \) we still use the GMRES(m) method for (18).

4. Results

In reality people care a lot about the result of the optimal exercise boundary \( \mathcal{S}(\tau) \), because it is a crucial factor in which the investors are used to decide whether to keep on holding agreement or just execute it. As a result, it is of great importance to obtain a relatively accurate curve describing the boundary \( \mathcal{S}(\tau) \) since this optimal exercise boundary \( \mathcal{S}(\tau) \) can not be fixed in advance. Figure 1 gives a graphical display of the optimal exercise boundaries. Note that the data are obtained with the SDA method and have not been smoothed. The curve in Figure 1 is smoother than in [11] and [19] (both obtained by other methods). Especially near the original points and \( \mathcal{S}(\tau) \) in Figure 1, the results are very close to those obtained by formula (5) when \( \tau = T - t \to 0 \).

In addition, implicit finite difference method was applied in paper [7], Compared to paper [7], in our approach, explicit finite difference method has applied, and the values need not to be corrected according to experiential formula, and the results are entirely analytic values. On the side, Figure 2 offers a graphical display of the option values.
$X = 1.0$, $h = 1/(M + 1)$, $M = 40$, $r = 0.6$, $\delta = 0.4$, $S = 1.0$

![Graph of optimal exercise boundary]

Time $\tau = T - t$ ($T = 0.6$, $N = 2521$, $k = T/(N + 1)$)

Fig. 1. Optimal exercise boundary $S(\tau)$ with SDA method

$X = 1.0$, $h = 1/(M + 1)$, $M = 40$, $r = 0.6$, $\delta = 0.4$, $S = 1.0$

![Graph of option values]

Time $\tau = T - t$ ($T = 0.6$, $N = 2521$, $k = T/(N + 1)$)

Fig. 2. Option values at different time with SDA method
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