Chapter 8
Conduction with Internal Energy Generation

1 Introduction

Many important problems are encountered in which heat conduction takes place in a medium inside which energy is being generated by some process. The process in question may the passage of electical current, a chemical reaction or a nuclear reaction. Conduction in systems with internal energy generation can take place under transient conditions, like in many induction heating applications or under steady state conditions as in nuclear engineering applications. The basics of heat conduction analysis with internal energy generation are examined below.

2 The Heat Equation for Systems with Internal Energy Generation

As described before the thermal differential energy balance equation describing the conduction of heat in a solid undergoing internal heat generation can be expressed as However, since the integrals are equal the arguments are also equal and the most general form of the differential thermal energy balance equation is

\[
\frac{\partial H}{\partial t} = -\nabla \cdot \mathbf{q} + g(r, t)
\]

and is called the heat equation. where \(H\) is the enthalpy, \(q\) is the heat flux, and \(g\) is the voluminic rate of internal energy generation. Under steady sate conditions, this reduces to

\[-\nabla \cdot \mathbf{q} + g(r, t) = 0\]

For a homogeneous material undergoing no phase changes and introducing the definitional relationship for the heat flux, the general equation above reduces to

\[
\rho C_p \frac{\partial T}{\partial t} = \nabla \cdot k\nabla T + g(r, t)
\]
\( \rho \) is the density, \( C_p \) is the specific heat, \( k \) is the thermal conductivity, and \( T \) is the temperature. While the steady state equation becomes
\[
\nabla \cdot k \nabla T + g(\mathbf{r}, t) = 0
\]

3 General Approach used to obtain Analytical Solutions

Simple problems can be readily solved by standard ordinary differential equation methods. For instance, the steady state heat conduction equation inside a cylinder (radius \( r_0 \)) made of homogeneous material of constant thermal properties undergoing internal energy generation at a constant rate in cylindrical polar coordinates with azimuthal symmetry and immersed in a medium that maintains the temperature at the surface of the cylinder at \( T_\infty \). The mathematical formulation of the problem is
\[
\frac{1}{r} \frac{d}{dr} \left( r k \frac{dT}{dr} \right) + g = 0
\]
subject to the boundary conditions
\[
\frac{dT}{dr} \bigg|_{r=0} = 0
\]
and
\[
T_{r_0} = T_\infty
\]
By direct integration, the desired solution is
\[
T(r) = T_\infty + \left( \frac{g}{k} \right)(r_0^2 - r^2)
\]
A generic approach to finding analytical solutions to conduction problems with internal heat generation consists in decomposing the original problem into two, easier to solve but otherwise equivalent problems.

Consider the non-homogeneous problem of transient multidimensional heat conduction with internal heat generation for a solid with constant properties
\[
\nabla^2 T(\mathbf{r}, t) + \frac{1}{k} g(\mathbf{r}) = \frac{1}{\alpha} \frac{\partial T(\mathbf{r}, t)}{\partial t}
\]
subject to non-homogeneous time independent conditions on at least part of the boundary.

This problem is now decomposed into a set of steady state non-homogeneous problems in each of which a single non-homogeneous boundary condition occurs, and a transient homogeneous problem. If the solutions of the steady state problems are \( T_{0j}(\mathbf{r}) \) and that for the transient problem is \( T_h(\mathbf{r}, t) \), the solution of the original problem is
\[
T(\mathbf{r}, t) = T_h(\mathbf{r}, t) + \sum_{j=0}^{N} T_{0j}(\mathbf{r})
\]
4 Finite Differences Methods

Finally consider now the problem of steady state conduction with internal heat generation

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{g(x, y)}{k} \]

on \((x, y)\) in the interior \(R = \{(x, y): a < x < b, c < y < d\}\) of a planar region and subject to

\[ T(x, y) = T_S(x, y) \]

along the boundary \(S\) of the region. As long as \(g\) and \(T_S\) are continuous a unique solution exists.

As before, a simple approach to the numerical solution of this problem consists in partitioning the intervals \([a, b]\) and \([c, d]\) respectively into \(N - 1\) and \(M - 1\) subintervals with step sizes \(\Delta x = (b - a)/(N - 1)\) and \(\Delta y = (d - c)/(M - 1)\) so that the node or mesh point \((x_i, y_j)\) is at \(x_i = a + (i - 1)\Delta x\) for \(i = 1, ..., N\) and \(y_j = c + (j - 1)\Delta y\) for \(j = 1, ..., M\). Using now second order accurate centered finite difference approximations for the partial derivatives the following five point formula for the temperature at node \((x_i, y_j)\) is obtained.

\[ -\frac{1}{\Delta x^2}T_{i-1,j} - \frac{1}{\Delta x^2}T_{i+1,j} - \frac{1}{\Delta y^2}T_{i,j-1} - \frac{1}{\Delta y^2}T_{i,j+1} + \left(\frac{2}{\Delta x^2} + \frac{2}{\Delta y^2}\right)T_{i,j} = \frac{g(x_i, y_j)}{k} \]

The boundary conditions imposed along the boundary nodes become

\[ T_{0,j} = T_S(x_1, y_j) \quad j = 1, 2, ..., M \]
\[ T_{N,j} = T_S(x_N, y_j) \quad j = 1, 2, ..., M \]
\[ T_{i,0} = T_S(x_i, y_1) \quad i = 1, 2, ..., N \]
\[ T_{i,M} = T_S(x_i, y_M) \quad i = 1, 2, ..., N \]

This is a system of simultaneous algebraic equations. To obtain a banded matrix the mesh points must be relabeled sequentially from left to right and from top to bottom. The resulting system can be solved by Gaussian elimination if \(N\) and \(M\) are small and by SOR iteration when they are large.

5 Finite Volume Method

Consider the problem steady state conduction in a slab with internal heat generation. The problem consists of steady state heat transport through the thickness \(L\) of a large slab with
constant internal heat generation \( g \) is described by the following form of the heat or diffusion equation

\[
\frac{d}{dx}(k \frac{dT}{dx}) + g = 0
\]

To implement the finite volume method first subdivide the thickness of the slab into a collection of \( N \) adjoining segments of thickness not necessarily of uniform size (finite volumes) and place a node inside each volume. Thus, an arbitrary node will be called \( P \), its size is \( \Delta x \) and the nodes to its left and right, respectively \( W \) and \( E \).

Note that two different types of nodes result. While \textit{interior} nodes are surrounded by finite volume on both sides, \textit{boundary} nodes contain material only on one side. Here we shall concentrate on the derivation of discrete equations for the interior nodes. Those for the boundary nodes will be discussed later.

The distance between nodes \( W \) and \( P \) is \( \delta x_w \) and that between \( P \) and \( E \), \( \delta x_e \). The locations of the finite volume boundaries corresponding to node \( P \) will be denoted by \( w \) and \( e \). Finally, the distance between \( P \) and \( e \) is called \( \delta x_{e-} \) and that between \( e \) and \( E \) is \( \delta x_{e+} \). If \( P \) is located in the center of the finite volume then \( \delta x_{e-} = \delta x_{e+} = \frac{1}{2} \delta x_e \).

To implement the finite volume method we now integrate the above equation over the span of the finite volume associated with node \( P \), i.e. from \( x_w \) to \( x_e \)

\[
\int_{x_w}^{x_e} d(k \frac{dT}{dx}) + \int_{x_w}^{x_e} gdx = (k \frac{dT}{dx})_e - (k \frac{dT}{dx})_w + g\Delta x = 0
\]

Next, approximate the derivatives by piecewise linear profiles to give

\[
k_e \frac{T_E - T_P}{\delta x_e} - k_w \frac{T_P - T_W}{\delta x_w} + g\Delta x = 0
\]

where the conductivities at the finite volume faces are calculated as the harmonic means of the values at the neighboring nodes, i.e.

\[
k_e = \left[ 1 - \frac{\delta x_{e+}/\delta x_e}{k_P} + \frac{\delta x_{e+}/\delta x_e}{k_E} \right]^{-1}
\]

and

\[
k_w = \left[ 1 - \frac{\delta x_{w+}/\delta x_w}{k_W} + \frac{\delta x_{w+}/\delta x_w}{k_P} \right]^{-1}
\]

Rearranging one obtains the algebraic equation

\[
a_P T_P = a_E T_E + a_W T_W + b
\]

where the coefficients are given by

\[
a_E = \frac{k_e}{\delta x_e}
\]
\[ a_W = \frac{k_w}{\delta x_w} \]

\[ a_P = a_E + a_W \]

and

\[ b = g \Delta x \]

One algebraic equation like the above, relating the values of \( T \) at three contiguous nodes, is obtained for each of the \( N \) nodes. Adding the this set the discrete equations associated with the boundary nodes one obtains a consistent set of interlinked simultaneous algebraic equations which must be solved to give the values of \( T \) for all nodal locations.

For the special case of constant thermal properties and finite volumes of uniform size \((\delta x_e = \delta x_w = \Delta x)\), the above is easily rearranged as

\[
\frac{T_E - 2T_P + T_W}{\Delta x^2} + \frac{g}{k} = 0
\]

which coincides with the FD formula obtained before.

### 6 Finite Element Method

Consider the boundary value problem of steady state heat conduction inside a slab \((L = 1)\), which is maintained at zero temperature on both its outside surfaces and that undergoes internal heat generation at a position dependent rate \( g = 20x^3 \). The governing equation is

\[
-\frac{d^2T}{dx^2} = 20x^3
\]

subject to the boundary conditions \( T(0) = T(1) = 0 \).

Now we find an approximate solution using the Galerkin finite element method with three equal size elements and the following global finite element basis functions:

\[
\phi_1(x) = \begin{cases} 
1 - 3x & \text{for } x \in [0, 1/3] \\
0 & \text{elsewhere}
\end{cases}
\]

\[
\phi_2(x) = \begin{cases} 
3x & \text{for } x \in [0, 1/3] \\
2 - 3x & \text{for } x \in [1/3, 2/3] \\
0 & \text{elsewhere}
\end{cases}
\]
\[\phi_3(x) = \begin{cases} 3x - 1 & \text{for } x \in [1/3, 2/3] \\ 3 - 3x & \text{for } x \in [2/3, 1] \\ 0 & \text{elsewhere} \end{cases}\]

\[\phi_4(x) = \begin{cases} 3x - 2 & \text{for } x \in [2/3, 1] \\ 0 & \text{elsewhere} \end{cases}\]

The variational representation of the problem is readily obtained by first multiplying the given differential equation by a function \(v\) and subsequently integrating from \(x = 0\) to \(x = 1\). After integration by parts, the result is

\[a(T, v) = (f, v)\]

where

\[a(u, v) = \int_0^1 dT \frac{dv}{dx} dx\]

and

\[(f, v) = \int_0^1 20x^3 v dx\]

Now, let us proceed to the implementation of the finite element method. Let the finite element approximation be \(u_h(x) \approx T(x)\). Introduce now for simplicity three elements \(e_1, e_2, e_3\) and four nodes \(i = 1, 2, 3, 4\). The nodal values of \(u\) are \(u_i, i = 1, 2, 3, 4\) and since \(u\) is specified at \(x = 0\) and \(x = 1\), then \(u_1 = u_4 = 0\) and the only unknowns are \(u_2\) and \(u_3\). The representation of the trial function \(u\) in terms of the nodal values then becomes

\[u_h = \sum_{i=1}^{4} u_i \phi_i = u_2 \phi_2 + u_3 \phi_3 = [\phi_2, \phi_3] \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}\]

In the Galerkin method the test function \(v\) is given directly in terms of the basis functions. In the present case, since \(u_1 = u_4 = 0\), \(v\) is simply given by

\[v_h = \phi_2 + \phi_3 = [1, 1] \begin{bmatrix} \phi_2 \\ \phi_3 \end{bmatrix}\]

Substituting the above into the variational formulation yields

\[a(u_h, v_h) = [1, 1] \begin{bmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}\]

and

\[(f, v) = [1, 1] \begin{bmatrix} F_2 \\ F_3 \end{bmatrix}\]
where

\[ k_{22} = \int_0^1 (\frac{d\phi_2}{dx})^2 dx = \int_0^{2/3} (\frac{d\phi_2}{dx})^2 dx = 6 \]

\[ k_{23} = k_{32} = \int_0^1 \frac{d\phi_2}{dx} \frac{d\phi_3}{dx} dx = \int_{1/3}^{2/3} \frac{d\phi_2}{dx} \frac{d\phi_3}{dx} dx = -3 \]

\[ k_{33} = \int_0^1 (\frac{d\phi_3}{dx})^2 dx = \int_{1/3}^1 (\frac{d\phi_3}{dx})^2 dx = 6 \]

\[ F_2 = \int_0^1 20x^3 \phi_2 dx = \int_0^{1/3} 20x^3 3x dx + \int_{1/3}^{2/3} 20x^3 (2 - 3x) dx = \frac{4}{81} + \frac{20}{81} = \frac{10}{27} \]

and

\[ F_3 = \int_0^1 20x^3 \phi_3 dx = \int_{1/3}^{2/3} 20x^3 (3x - 1) dx + \int_0^{2/3} 20x^3 (3 - 3x) dx = \frac{49}{81} + \frac{131}{81} = \frac{60}{27} \]

Therefore, the finite element equation is

\[ Ku = F = \left[ \begin{array}{cc} 6 & -3 \\ -3 & 6 \end{array} \right] \left[ \begin{array}{c} u_2 \\ u_3 \end{array} \right] = \left[ \begin{array}{c} \frac{10}{27} \\ \frac{60}{27} \end{array} \right] \]

Finally, solving for \( u_2 \) and \( u_3 \) yields

\[ u_2 = \frac{80}{243} \]

\[ u_3 = \frac{130}{243} \]