Chapter 6  
Conduction in Composite Media

1 Introduction

Composites are formed by bringing together into close contact distinct materials. The material properties of the assembly change discontinuously at the contact surface. Also, the mechanical conditions of the contact at the interface affect the rate at which energy flows across it. In general, the analysis of heat transfer at an interphase interface characterized by its normal \( n \) and separating the two materials 1 and 2 requires specification of two conditions: a) the heat flux balance

\[
(\nabla T \cdot n)_1 = (\nabla T \cdot n)_2
\]

and the relationship between the temperatures on both sides of the interface \( T_1 \) and \( T_2 \) usually in terms of the contact conductance \( h \), i.e.

\[
(\nabla T \cdot n)_1 = h(T_1 - T_2)
\]

The so called perfect thermal contact condition simply states that at the interface \( h \rightarrow \infty \) and \( T_1 = T_2 \). The above equations must be added to the stated initial and boundary conditions of the particular problem at hand.

2 Mathematical Formulation of 1D Problems

Consider now a composite material formed by stacking \( M \) thin sheets \( i = 1, 2, 3, \ldots, M \) with thermal conductivities \( k_i \) and diffusivities \( \alpha_i \). Let the locations of the interfaces in the system be given by \( x_i \) with \( i = 1, 2, 3, \ldots, M+1 \) where interfaces 1 and \( M+1 \) are the outer boundaries of the composite. Let the contact conductances at interphase interfaces be \( h_i \) for \( i = 2, 3, \ldots, M \), the heat transfer coefficients at the outer boundaries \( h_1 \) and \( h_{M+1} \) and the temperatures of the surrounding environments \( T_{\infty,1}(t) \) and \( T_{\infty,M}(t) \). Finally let the initial temperature distribution within each layer be \( T_i(x,0) = F_i(x) \), the internal heat generation rate \( g_i(x,t) \), and the temperature after time \( t \) \( T_i(x,t) \), for \( i = 1, 2, 3, \ldots, M \).
The mathematical formulation of the problem consists of finding \( T_i(x, t) \) for all \( i \) which satisfy the transient nonhomogeneous 1D heat equation

\[
\alpha_i \frac{\partial}{\partial x} \left( \frac{\partial T_i}{\partial x} \right) + \frac{\alpha_i}{k_i} g_i(x, t) = \frac{\partial T_i}{\partial t}
\]

and the initial condition

\[
T_i(x, 0) = F_i(x)
\]

and subject to

\[
-k_1 \frac{\partial T_1}{\partial x} = h_1(T_{\infty, 1}(t) - T_1)
\]

at \( x = x_1 \),

\[
-k_M \frac{\partial T_M}{\partial x} = h_{M+1}(T_{\infty, M}(t) - T_M)
\]

at \( x = x_{M+1} \), and

\[
-k_i \frac{\partial T_i}{\partial x} = h_{i+1}(T_i - T_{i+1})
\]

and

\[
k_i \frac{\partial T_i}{\partial x} = k_{i+1} \frac{\partial T_{i+1}}{\partial x}
\]

at each of the contact surfaces \( i = 2, 3, ..., M \). These conditions at the contact interfaces represent a system of \( 2M \) equations which determine the values of the \( 2M \) interface temperatures \( T_i \) for \( i = 2, 3, ..., M \).

Note that the stated problem involves nonhomogeneous boundary conditions at \( x = x_1 \) and \( x = x_{M+1} \). In order to solve the problem it must be reformulated in terms of three simpler problems whose solutions are then superimposed to obtain the desired \( T_i(x, t) \). The associated simpler problems consist of two steady state problems without internal heat generation, each subject to a single nonhomogeneous boundary condition, and a transient problem with internal heat generation but subject to homogeneous boundary conditions.

Consider a composite formed by two layers in perfect thermal contact. Assume the temperature at the outer surface at \( x = x_1 = 0 \) is \( T_1(x_1, t) = f_1(t) \) while at \( x = x_3 = b \) there is convective exchange \((h_3)\) with a medium at \( T_{\infty, 2}(x, t) = f_2(t) \). The initial temperatures of the two materials are \( F_1(x) \) and \( F_2(x) \). The interphase interface is at \( x = x_2 = a \).

To reformulate the problem, the desired temperature distribution \( T_i(x, t), i = 1, 2 \) is expressed as

\[
T_i(x, t) = \phi_i(x) f_1(t) + \psi_i(x) f_2(t) + \theta_i(x, t)
\]
The functions $\phi_i(x)$ satisfy
\[ \phi''_1(x) = \phi''_2(x) = 0 \]
subject to
\[ \phi_1 = 1 \]
at $x = 0$,
\[ k_2 \phi'_2 + h_3 \phi_2 = 0 \]
at $x = b$ and
\[ \phi_1 = \phi_2 \]
and
\[ k_1 \phi'_1 = k_2 \phi'_2 \]
at $x = a$.

The functions $\psi_i(x)$ satisfy
\[ \psi''_1(x) = \psi''_2(x) = 0 \]
subject to
\[ \psi_1 = 0 \]
at $x = 0$,
\[ k_2 \psi'_2 + h_3 \phi_2 = h_3 \]
at $x = b$ and
\[ \psi_1 = \psi_2 \]
and
\[ k_1 \psi'_1 = k_2 \psi'_2 \]
at $x = a$.

Finally, the functions $\theta_i(x, t)$ satisfy
\[ \alpha_1 \frac{\partial^2 \theta_1}{\partial x^2} = \frac{\partial \theta_1}{\partial t} + g_1^* \]
in $0 < x < a$,
\[
\alpha_2 \frac{\partial^2 \theta_2}{\partial x^2} = \frac{\partial \theta_2}{\partial t} + g_2^*
\]
in $a < x < b$, where $g_i^* = -\phi_i(\frac{df_i}{dt}) - \psi_i(\frac{df_i}{dt})$, subject to
\[
\theta_1(0,t) = 0
\]
at $x = 0$,
\[
k_2 \frac{\partial \theta_2}{\partial x} + h_3 \theta_2 = 0
\]
at $x = b$, and with
\[
\theta_1 = \theta_2
\]
and
\[
k_1 \frac{\partial \theta_1}{\partial x} = k_2 \frac{\partial \theta_2}{\partial x}
\]
at $x = a$.

3 Orthogonal Expansion Technique

Consider the following transient 1D conduction problem in a $M$-layer composite sheet:
\[
\alpha_i \frac{\partial}{\partial x} \left( \frac{\partial T_i}{\partial x} \right) = \frac{\partial T_i}{\partial t}
\]
and the initial condition
\[
T_i(x,0) = F_i(x)
\]
and subject to
\[
-k_1 \frac{\partial T_1}{\partial x} + h_1 T_1 = 0
\]
at $x = x_1$,
\[
-k_M \frac{\partial T_M}{\partial x} + h_{M+1} T_M = 0
\]
at $x = x_{M+1}$, and
\[
-k_i \frac{\partial T_i}{\partial x} = h_{i+1}(T_i - T_{i+1})
\]
and
\[ k_i \frac{\partial T_i}{\partial x} = k_{i+1} \frac{\partial T_{i+1}}{\partial x} \]
at each of the contact surfaces \( i = 2, 3, ..., M \).

Now assume the required solution is of the form
\[ T_i(x, t) = \psi_i(x) \Gamma(t) \]
Substituting into the heat equation one gets
\[ \frac{\alpha_i}{\psi_{i,n}} \psi_{i,n}'' = \frac{1}{\Gamma'} = -\beta_n^2 \]
where \( \beta_n \) and \( \psi_{i,n} \) are, respectively, the eigenvalues and eigenfunctions of the problem. The boundary conditions in turn become
\[ -k_1 \psi_{1,n}'' + h_1 \psi_{1,n} = 0 \]
at \( x = x_1 \),
\[ -k_M \psi_{M,n}'' + h_{M+1} \psi_{M,n} = 0 \]
at \( x = x_{M+1} \), and
\[ -k_i \psi_{i,n} = h_{i+1}(\psi_{i,n} - \psi_{i+1,n}) \]
and
\[ k_i \psi_{i,n} = k_{i+1} \psi_{i+1,n} \]
at each of the contact surfaces \( i = 2, 3, ..., M \).

The required orthogonality relationship is
\[ \sum_{i=1}^{M} \frac{k_i}{\alpha_i} \int_{x_i}^{x_{i+1}} \psi_{i,n}(x) \psi_{i,r}(x) dx = \begin{cases} 0; & n \neq r \\ N_n = \sum_{j=1}^{M} \frac{k_j}{\alpha_j} \int_{x_j}^{x_{j+1}} \psi_{j,n}(x)^2 dx; & n = r \end{cases} \]

Since \( \Gamma(t) = e^{-\beta_n^2 t} \), the required solution is
\[ T_i(x, t) = \sum_{n=1}^{\infty} c_n e^{-\beta_n^2 t} \psi_{i,n}(x) \]
for \( i = 1, 2, ..., M \). This solution must also satisfy the initial condition so, substituting and operating both sides with \( \int_{x_i}^{x_{i+1}} \psi_{i,r}(x) dx \) leads to
\[ c_n = \frac{1}{N_n} \sum_{j=1}^{M} \frac{k_j}{\alpha_j} \int_{x_j}^{x_{j+1}} \psi_{j,n}(x) F_j(x) dx \]
Substitution of this expression into the equation above it leads to the required solution. Note that if \( M = 1 \) the solution reduces to the solution given earlier for a single phase material.

The eigenfunctions \( \psi_{i,n} \) are given by

\[
\psi_{i,n}(x) = A_{i,n} \phi_{i,n}(x) + B_{i,n} \theta_{i,n} = A_{i,n} \sin\left(\frac{\beta_n}{\sqrt{\alpha_i}} x\right) + B_{i,n} \cos\left(\frac{\beta_n}{\sqrt{\alpha_i}} x\right)
\]

The corresponding functions \( \phi_{i,n} \) and \( \theta_{i,n} \) for cylindrical coordinates are \( J_0(\beta_n x / \sqrt{\alpha_i}) \) and \( Y_0(\beta_n x / \sqrt{\alpha_i}) \), respectively. Further, the corresponding functions \( \phi_{i,n} \) and \( \theta_{i,n} \) for spherical coordinates are \((1/x) \sin(\beta_n x / \sqrt{\alpha_i})\) and \(1/x \cos(\beta_n x / \sqrt{\alpha_i})\), respectively.

Finally, the coefficients \( A_{i,n} \) and \( B_{i,n} \) and the eigenvalues \( \beta_n \) are determined from the 2\( M \) contact interface conditions. The coefficients are determined directly from the system of equations formed by the contact conditions. The eigenvalues are determined from the requirement that for a nontrivial solution the determinant of the coefficients \( A_{i,n} \) and \( B_{i,n} \) must vanish. For instance, for a composite sheet consisting of two layers 1 and 2 with \( 0 \leq x \leq a \) and \( a \leq x \leq b \) the eigenvalues are defined as the roots of

\[
\begin{vmatrix}
\sin \gamma & -\sin(\frac{\alpha}{\eta}) & -\cos(\frac{\beta}{\eta}) \\
K \cos \gamma & -\cos(\frac{\alpha}{\eta}) & \sin(\frac{\beta}{\eta}) \\
0 & \frac{K}{\eta} \sin \eta + \cos \eta & \frac{H}{\eta} \cos \eta - \sin \eta
\end{vmatrix} = 0
\]

where \( \gamma = a \beta_n / \sqrt{\alpha_1} \), \( \eta = b \beta_n / \sqrt{\alpha_2} \), \( H = bh_3/k_2 \) and \( K = (k_1/k_2) \sqrt{\alpha_1/\alpha_2} \). And for a composite cylinder consisting of two layers 1 and 2 with \( 0 \leq r \leq a \) and \( a \leq r \leq b \) the eigenvalues are defined as the roots of

\[
\begin{vmatrix}
J_0(\gamma) & J_0(\frac{a}{\eta}) & -Y_0(\frac{a}{\eta}) \\
KJ_1(\gamma) & J_1(\frac{a}{\eta}) & -Y_1(\frac{a}{\eta}) \\
0 & \frac{H}{\eta} J_0(\eta) - J_1(\eta) & \frac{H}{\eta} Y_0(\eta) - Y_1(\eta)
\end{vmatrix} = 0
\]