Session 6
Heat Conduction in Cylindrical and Spherical Coordinates II

1 Introduction

Heat conduction problems in cylindrical and spherical coordinates are readily solved using numerical methods. Finite difference, finite volume and finite element methods can all be applied.

2 Finite Difference Methods

Consider the problem of transient heat conduction in a solid cylinder of radius $R$ with azimuthal symmetry and independent of $z$

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r} \frac{\partial}{\partial r} (r \frac{\partial T}{\partial r})$$

Introducing a mesh of $N$ nodes along the $r-$direction, $r_i$ with $i = 1, 2, ..., N$ and $\Delta r = R/(N-1)$ and a mesh of nodes in time $t_j$, with $j = 1, 2, ...,$ spacing $\Delta t$, and forward differencing in time, a finite difference analog is

$$\frac{T_{i,j+1} - T_{i,j}}{\Delta t} = \alpha \frac{r i+1/2 \left(\frac{T_{i+1,j} - T_{i,j}}{\Delta r}\right) - r i-1/2 \left(\frac{T_{i,j} - T_{i-1,j}}{\Delta r}\right)}{r_i \Delta r}$$

where $r_{i+1/2}$ is a radial position located halfway between $r_{i+1}$ and $r_i$, $r_{i-1/2}$ is a radial position located halfway between $r_i$ and $r_{i-1}$ and $T_{i,j} \approx T(r_i, t_j)$. This constitutes an explicit method for the direct determination of the unknown temperatures at all nodes at the new time level $n+1$.

If backward differencing in time is used instead, the result is

$$\frac{T_{i,j+1} - T_{i,j}}{\Delta t} = \alpha \frac{r i+1/2 \left(\frac{T_{i+1,j+1} - T_{i,j+1}}{\Delta r}\right) - r i-1/2 \left(\frac{T_{i,j+1} - T_{i-1,j+1}}{\Delta r}\right)}{r_i \Delta r}$$
Since one equation is obtained for each node and each equation relates the approximate value of \( T \) at the node with those of its two neighboring nodes one has then an \textit{implicit scheme}. The result is a system of interlinked simultaneous algebraic equations with simple tridiagonal structure which is readily solved using standard numerical linear algebra methods.

As a second example consider the problem of transient heat conduction in a solid sphere of radius \( R \) with azimuthal and poloidal symmetry

\[
\frac{\partial T}{\partial t} = \frac{\alpha}{r^2} \left[ \frac{\partial}{\partial r} (r^2 \frac{\partial T}{\partial r}) \right]
\]

Introducing a mesh of \( N \) nodes along the \( r \)–direction, \( r_i \) with \( i = 1, 2, ..., N \) and \( \Delta r = R/(N - 1) \) and a mesh of nodes in time \( t_j \), with \( j = 1, 2, ..., \), spacing \( \Delta t \), and forward differencing in time, a finite difference analog is

\[
\frac{T_{i,j+1} - T_{i,j}}{\Delta t} = \frac{\alpha}{r_{i+1/2}} \left( \frac{T_{i+1,j} - T_{i,j}}{\Delta r} \right) - \frac{\alpha}{r_{i-1/2}} \left( \frac{T_{i,j} - T_{i-1,j}}{\Delta r} \right)
\]

where \( T_{i,j} \approx T(r_i, t_j) \). Again, this is an explicit scheme.

If backward differencing in time is used instead, the result is

\[
\frac{T_{i,j+1} - T_{i,j}}{\Delta t} = \frac{\alpha}{r_{i+1/2}} \left( \frac{T_{i+1,j+1} - T_{i,j+1}}{\Delta r} \right) - \frac{\alpha}{r_{i-1/2}} \left( \frac{T_{i,j+1} - T_{i-1,j+1}}{\Delta r} \right)
\]

Once again, an implicit system of interlinked simultaneous algebraic equations with simple tridiagonal structure is obtained.

As in the Cartesian coordinates case, the explicit method is conditionally stable and the time step must be selected so as to satisfy the CFL condition, namely

\[
\Delta t < \frac{\Delta r^2}{2\alpha}
\]

3 Boundary Conditions

When a multidimensional problem exhibits cylindrical or spherical symmetry it can be represented as a one dimensional problem in polar coordinates. For example for radially symmetric heat conduction in rods or spheres

\[
\frac{\partial T}{\partial t} = \frac{\alpha}{r^\gamma} \frac{\partial}{\partial r} (r^\gamma \frac{\partial T}{\partial r}) = T_t = \alpha [T_{rr} + \frac{\gamma}{r} T_r]
\]

where \( \gamma = 1 \) for systems with cylindrical symmetry and \( \gamma = 2 \) for systems with spherical symmetry. Subscript notation for derivatives is used for simplicity of representation.
All ideas presented before can be directly applied here with little modification. However, special care is required to handle the symmetry condition at the origin \( r = 0 \). For symmetry, it is required that

\[
\frac{\partial T}{\partial r} = 0
\]

By means of a Maclaurin expansion one can show that at the origin, the following form of the heat equation is valid (when one has symmetry at the origin),

\[
T_t = (\gamma + 1)\alpha T_{rr}
\]

Introducing a phantom node next to the origin, as in the Cartesian case and expressing the above two equations in terms of their finite difference analoges it is possible to obtain a finite difference formula for the node point located at the origin.

If no symmetry can be assumed the following expressions can be used instead to approximate the Laplacian,

\[
\nabla^2 T \approx \frac{4(T_{M,j} - T_{0,j})}{\Delta r^2}
\]

for cylindrical systems and

\[
\nabla^2 T \approx \frac{6(T_{M,j} - T_{0,j})}{\Delta r^2}
\]

for spherical systems. Here \( T_{M,j} \) is the nearest-neighbor mean value of \( T \) obtained by averaging over all nearest neighbor nodes to the origin. The above approximations can then be used together with the original heat equation

\[
\frac{\partial T}{\partial t} = \alpha \nabla^2 T
\]

at the origin in order to obtain finite difference formulae for \( T_{0,j} \).