Viscoelasticity
Mechanical properties of living tissues

F7  Soft tissues
F8  Muscles
F9  Viscoelasticity
F15 Bone
Lab

- Construct a Finite element Model
- Dissection of deer spine
- Mechanical testing
- Analysis of the results and implement properties into FE-model
- Report

<table>
<thead>
<tr>
<th>Laboration</th>
<th>Group number</th>
<th>Date</th>
<th>Time</th>
<th>Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>week 16</td>
<td>AN/DL</td>
<td>Fri</td>
<td>18/4</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>8-12</td>
<td></td>
<td>Dissec + mech test</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1-4</td>
<td></td>
<td>AN/DL</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Fri</td>
<td>18/4</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>13-17</td>
<td></td>
<td>Dissec + mech test</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5-8</td>
<td></td>
<td>AN/DL</td>
</tr>
</tbody>
</table>
Viscoelasticity

Material from:

Kleiven S. An Introduction to Viscoelasticity, Lecture notes for 4E1150,

Fung, Y.C. Foundations of Solid Mechanics, Prentice-Hall Inc.,

Viscoelasticity

- Characteristic behaviour of viscoelastic materials
- Why are biological tissues viscoelastic?
- Theory of viscoelasticity
Linear Viscoelasticity

• Soft tissues exhibit several Viscoelastic properties:
  – *stress relaxation* at constant strain
  – *creep* at constant stress
  – *hysteresis* during loading and unloading
  – *strain-rate* dependence

• i.e., in general, stress in soft tissues depends on strain and the *history* of strain

• These properties can be modeled by the theory of *viscoelasticity*
Stress relaxation

Steel bolt through plastic...

The load will decrease in time for a constant applied deformation

Time

Load
Stress relaxation
Examples of Biological tissues in Stress relaxation

Stress relaxation function for Nucleus Pulposus (Normalized) (Iatridis et al., 1997)

Stress relaxation curves for the Scalp (Melvin et al., 1970)
Creep

Structure loaded by gravity…

Deformation

mg

Deformation will continue in time although the load is constant

Time
Creep

Example of Biological tissues in Creep

Creep compliance of dura mater in tension
(Galford et al., 1970)
Hysteresis and strain-rate dependence

Area between curves is proportional to dissipated energy

Increased strain-rate leads to a stiffer response
Hysteresis and strain-rate dependence

Example of hysteresis & strain-rate dependence in Biological tissues

Hysteresis loops for ankle ligament
(Funk et al., 2000)

The stress-strain relationship for bone
(McElhaney, 1966)
Viscoelasticity

- Characteristic behaviour of viscoelastic materials
- Why are biological tissues viscoelastic?
- Theory of viscoelasticity
Ligaments & Tendons

Rate dependent strength, stiffness and energy absorption.
Increased by a factor of 3 if the load rate is increased from 8 to 2300 m/s.

**Viscoelastic** response due to
- Interactions between the proteoglycans in the groundsubstance and collagen fibrils.
Cartilage – low friction layer in the joints

- Damping between two bones, for example in the knee.
- 70-80% water, collagen fibers and ground substance.

Viscoelastic response due to
- Fluid flow during loading
The intervertebral disc

- **Nucleus pulposus (NP)**
  - Hydrophilic gel
  - 90% water (decreases with age to 70%).

- **Annulus fibrosus (AP)**
  - Composite of collagen fibers in a ground substance.
  - Approximately 90 unidirectional laminae
  - Fiber direction ± 60°
  - 78% water

Viscoelastic response due to
- Fluid flow during loading
- Shear forces between the matrix and fibers during fiber straightening.
Trabecular bone

low-density, open cell, rod-type structure from the femur head

higher density, roughly prismatic cells from the femur head

intermediate density parallel plate structure with rods normal to the plate

Viscoelastic response due to
- Fluid flow during loading
Importance of viscoelasticity

- Structure made of polymers (i.e. FRP) -> Creep, relaxation etc....
  -> Collapse?
- Damping, dynamic modulus etc. of Biological tissues
  -> Protection!
Importance of viscoelasticity

Maximal shear strain for human head including viscoelasticity

Maximal shear strain for human head excluding viscoelasticity
Viscoelasticity

- Characteristic behaviour of viscoelastic materials
- Why are biological tissues viscoelastic?
- Theory of viscoelasticity
Basic elements

Spring (Hooke element)

\[ \sigma_{E_1} = E_1 \cdot \varepsilon \]

Viscous Damper (Newton element)

\[ \sigma_\eta = \eta \cdot \dot{\varepsilon} \]
Simple Viscoelastic Models

Stress depends on strain and strain-rate:

- Elastic stress depends on strain (spring)
- Viscous stress depends on strain-rate (damper)
  - Maxwell: Strains add in series, stresses are equal
  - Kelvin: Stresses add in parallel, strains are equal
  - SLS: Combination of Maxwell and Kelvin
Maxwell Model

Total strain = spring strain + dashpot strain:

\[ \varepsilon = \varepsilon_{E_1} + \varepsilon_\eta \]

\[ \sigma = \sigma_{E_1} = \sigma_\eta \]

\[ \sigma_{E_1} = E_1 \cdot \varepsilon_{E_1} \]

\[ \sigma_\eta = \eta \cdot \dot{\varepsilon}_\eta \]

\[ \Rightarrow \dot{\varepsilon}_{E_1} = \frac{\dot{\sigma}}{E_1} \]

\[ \dot{\varepsilon}_\eta = \frac{\sigma}{\eta} \]

A linear first-order ordinary differential equation (ODE)

\[ \dot{\varepsilon} = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta} \]
Does the model creep?

\[ \dot{\varepsilon} = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta} \]

Integrating, for constant applied stress, \( \sigma_0 \):

\[ \int_{\varepsilon(0)}^{\varepsilon(t)} d\varepsilon = \frac{1}{E_1} \int_{\sigma(0)}^{\sigma(t)} d\sigma + \frac{\sigma_0}{\eta} \int_{0}^{t} dt + C \Rightarrow \varepsilon(t) = \varepsilon(0) + \frac{\sigma_0 t}{\eta} + C \]
\[ \varepsilon(t) = \varepsilon(0) + \frac{\sigma_0 t}{\eta} + C \]

Only the Hooke element reacts initially: \[ \varepsilon(0) = C = \frac{\sigma_0}{E_1} \]

\[ \Rightarrow \varepsilon(t) = \frac{\sigma_0}{E_1} + \frac{\sigma_0 t}{\eta} = \sigma_0 \left( \frac{1}{E_1} + \frac{t}{\eta} \right) = \sigma_0 J(t) \]

Creep function
Relaxation Solution

Does the model relax?

A constant strain $\varepsilon_0$ is instantaneously applied at time $t=0$, when $\sigma=0$
Constant strain $\Rightarrow \frac{d\varepsilon}{dt}=0$, when $t>0$

\[
\frac{d\varepsilon}{dt} = \frac{1}{E_1} \frac{d\sigma}{dt} + \frac{\sigma}{\eta} = 0
\]

\[
\frac{1}{E_1} d\sigma + \frac{\sigma}{\eta} dt = 0 \Rightarrow \frac{d\sigma}{\sigma} = -\frac{E_1}{\eta} dt
\]

Integrating:

\[
\int_{\sigma(0)}^{\sigma(t)} \frac{d\sigma}{\sigma} = -\frac{E_1}{\eta} \int_0^t dt + C \Rightarrow \ln(\sigma(t)) - \ln(\sigma(0)) = -\frac{E_1}{\eta} t + C
\]
Relaxation Solution

\[ e^{\ln(\sigma(t)) - \ln(\sigma(0))} = e^{\left(-\frac{E_1}{\eta} t + C\right)} \Rightarrow \varepsilon_t^\f = C_1 \]

\[ \Rightarrow \frac{\sigma(t)}{\sigma(0)} = C_1 e^{-\frac{E_1}{\eta} t} \Rightarrow \varepsilon_t^\f(0) = E_1 \varepsilon_0 \rightarrow C_1 = 1 \]

(Only the Hooke element reacts initially):

\[ \Rightarrow \sigma(t) = \varepsilon_0 E_1 e^{-\frac{E_1}{\eta} t} = \varepsilon_0 E(t) \]

Relaxation function
Total stress = spring stress + dashpot stress:

\[ \sigma = \sigma_{E_1} + \sigma_\eta \]

\[ \varepsilon = \varepsilon_{E_1} = \varepsilon_\eta \]

\[ \sigma_{E_1} = E_1 \cdot \varepsilon \]

\[ \sigma_\eta = \eta \cdot \dot{\varepsilon} \]

\[ \sigma = E_1 \cdot \varepsilon + \eta \cdot \dot{\varepsilon} \]

A linear first-order ordinary differential equation (ODE)
\[ \sigma = E_1 \cdot \varepsilon + \eta \cdot \dot{\varepsilon} \]

**Creep Solution**

Does the model creep?

Constant stress, \( \sigma_0 \):

\[
\frac{d \varepsilon}{dt} + \frac{E_1}{\eta} \varepsilon = \frac{\sigma_0}{\eta} \quad \varepsilon(t) = \varepsilon_H(t) + \varepsilon_N(t)
\]

\[
\frac{d \varepsilon_H}{dt} + \frac{E_1}{\eta} \varepsilon_H = 0 \quad \Rightarrow \quad \frac{d \varepsilon_H}{\varepsilon_H} = -\frac{E_1}{\eta} dt \quad \Rightarrow \quad \varepsilon_H(t) = C_1 e^{-\frac{E_1}{\eta} t}
\]

\[
\varepsilon_N(t) = C_2 \quad \Rightarrow \quad \varepsilon(t) = C_1 e^{-\frac{E_1}{\eta} t} + C_2 \quad \Rightarrow \quad \frac{d \varepsilon(t)}{dt} = -\frac{E_1}{\eta} C_1 e^{-\frac{E_1}{\eta} t}
\]

\[
-\frac{E_1}{\eta} C_1 e^{-\frac{E_1}{\eta} t} + \frac{E_1}{\eta} C_1 e^{-\frac{E_1}{\eta} t} + \frac{E_1}{\eta} C_2 = \frac{\sigma_0}{\eta} \quad \Rightarrow \quad C_2 = \frac{\sigma_0}{E_1}
\]

\[
\Rightarrow \varepsilon(t) = C_1 e^{-\frac{E_1}{\eta} t} + \frac{\sigma_0}{E_1}
\]
\[ \varepsilon(0) = 0 \implies C_1 = -\frac{\sigma_0}{E_1} \implies \varepsilon(t) = \frac{\sigma_0}{E_1} (1 - e^{-\frac{E_1 t}{\eta}}) = \sigma_0 J(t) \]
Relaxation Solution

Does the model relax?

\[ \sigma = E_1 \cdot \varepsilon + \eta \frac{d\varepsilon}{dt} \]

An instantaneous change in the strain \(d\varepsilon = \varepsilon_0\) and \(dt \to 0\) gives \(\sigma(0) \to \infty\)

\[ \sigma(0) \to \infty \]

\[ \sigma(t > 0) = E_1 \cdot \varepsilon_0 \]
Standard Linear Solid

\[ \sigma = \sigma_{E_2} = \sigma_{\text{Kelvin}} \]

\[ \varepsilon = \varepsilon_{E_2} + \varepsilon_{\text{Kelvin}} \]

\[ \varepsilon_{E_2} = \frac{\sigma}{E_2} \quad \Rightarrow \quad \dot{\varepsilon}_{E_2} = \frac{\dot{\sigma}}{E_2} \]

\[ \sigma_{\text{Kelvin}} = E_1 \cdot \varepsilon_{\text{Kelvin}} + \eta \cdot \dot{\varepsilon}_{\text{Kelvin}} \]

\[ \Rightarrow \varepsilon_{\text{Kelvin}} = \frac{(\sigma - E_1 \cdot \varepsilon_{\text{Kelvin}})}{\eta} \]
Standard Linear Solid

\[ \Rightarrow \dot{\varepsilon} = \frac{\dot{\sigma}}{E_2} + \frac{(\sigma - E_1 \cdot \varepsilon_{kelvin})}{\eta} \]

\[ \varepsilon_{kelvin} = \varepsilon - \varepsilon_{E_2} \Rightarrow \]

\[ \dot{\varepsilon} = \frac{\dot{\sigma}}{E_2} + \frac{\sigma}{\eta} - \frac{E_1}{\eta} \cdot (\varepsilon - \varepsilon_{E_2}) \Rightarrow \]

\[ \varepsilon_{E_2} = \frac{\sigma}{E_2} \Rightarrow \]

\[ \dot{\varepsilon} = \frac{\dot{\sigma}}{E_2} + \frac{\sigma}{\eta} (1 + \frac{E_1}{E_2}) - \frac{E_1}{\eta} \varepsilon \]
Creep Solution

Does the model creep?

\[
\dot{\varepsilon} = \frac{\dot{\sigma}}{E_2} + \frac{\sigma}{\eta} \left(1 + \frac{E_1}{E_2}\right) - \frac{E_1}{\eta} \varepsilon
\]

A constant stress, \( \sigma_0 \) is instantaneously applied at time \( t=0 \),
Constant stress \( \Rightarrow d\sigma/dt=0 \), when \( t>0 \)

\[
\frac{d\varepsilon}{dt} + \frac{E_1}{\eta} \varepsilon = \frac{\sigma_0}{\eta} \left(1 + \frac{E_1}{E_2}\right) \Rightarrow
\]

A linear first-order ordinary differential equation (ODE)

Solving, and using the fact that only the Hooke element reacts initially

\[
\Rightarrow \varepsilon(0) = \frac{\sigma_0}{E_2} \Rightarrow
\]
Creep Solution

\[ \varepsilon(t) = \sigma_0 \left[ \frac{1}{E_1} (1 - e^{-\frac{E_1 t}{\eta}}) + \frac{1}{E_2} \right] = \sigma_0 J(t) \]

\[ \varepsilon(\infty) = \sigma_0 \left( \frac{1}{E_1} + \frac{1}{E_2} \right) \]
Relaxation Solution

Does the model relax?

\[
\dot{\varepsilon} = \frac{\dot{\sigma}}{E_2} + \frac{\sigma}{\eta} \left(1 + \frac{E_1}{E_2}\right) - \frac{E_1}{\eta} \varepsilon
\]

A constant strain \( \varepsilon_0 \) is instantaneously applied at time \( t=0 \), when \( \sigma=0 \), Constant strain \( \Rightarrow \frac{d\varepsilon}{dt}=0 \), when \( t>0 \)

\[
0 = \frac{\dot{\sigma}}{E_2} + \frac{\sigma}{\eta} \left(1 + \frac{E_1}{E_2}\right) - \frac{E_1}{\eta} \varepsilon_0 \Rightarrow
\]

\[
\dot{\sigma} + \frac{(E_1 + E_2)}{\eta} \sigma = \frac{E_1 E_2}{\eta} \varepsilon_0
\]

A linear first-order ordinary differential equation (ODE)

Solving, and using the initial conditions...
Relaxation Solution

\[ \sigma(t) = \varepsilon_0 \left[ \frac{E_2}{E_1 + E_2} e^{-\frac{E_1 + E_2}{\eta} t} + \frac{E_1 E_2}{E_1 + E_2} \right] = \varepsilon_0 E(t) \]

Instantaneous modulus \( E_0 \)

Asymptotic modulus \( E_\infty \)

\[ E(t) = E_\infty + (E_0 - E_\infty) e^{-\beta t} \]
Assume:\ \ \ \ \ \ \ \ \ \ \ \approx 0.45

In LS-DYNA

\[ G(t) = G_\infty + (G_0 - G_\infty)e^{-\beta t} \]

Use \( E_\infty \) to determine bulk modulus \( K \)

\[ E(t) = E_\infty + (E_0 - E_\infty)e^{-\beta t} \]
Linear Viscoelastic Models, Creep Functions

Maxwell

Kelvin

SLS
Linear Viscoelastic Models, Relaxation Functions

Maxwell

Kelvin

SLS

\[ \sigma(t) \]

\[ \varepsilon_0 \]

\[ t \]
Linear Viscoelasticity, Summary of Key Points

• In *viscoelastic* materials stress depends on strain and *strain-rate*
• They exhibit *creep, relaxation* and *hysteresis*
• Viscoelastic models can be derived by combining *springs with dampers, 3-parameter linear models* (e.g. SLS) have exponentially decaying creep and relaxation functions; *time constants* are the ratio of elasticity to damping
• The *instantaneous modulus, $E_0$*, is the stress-strain ratio at $t=0$
• The *asymptotic modulus, $E_\infty$*, is the stress-strain ratio as $t \to \infty$
What if the curve of the model does not fit the curve of the material we want to describe?

Generalized Models...
Generalized Kelvin Model

Total strain = \( \sum (\text{strains in each Kelvin element}) \)

\[
\varepsilon = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n = \sum_{i=1}^{n} \varepsilon_i
\]

Creep function:

\[
\varepsilon(t) = \sum_{i=1}^{n} \frac{\sigma_0}{E_i} \left(1 - e^{-\frac{E_{i,t}}{\eta_i}}\right)
\]
Generalized Maxwell Model

Total stress: \( \sigma = \sigma_1 + \sigma_2 + \ldots + \sigma_n = \sum_{i=1}^{n} \sigma_i \)

Relaxation function:

\[ \sigma(t) = \sum_{i=1}^{n} \varepsilon_0 E_i e^{-\frac{E_i t}{\eta_i}} \]
The Convolution Integral for Stress

\[ \Delta \sigma_2 = \Delta \varepsilon_2 E(t - t_2) \]
\[ \Delta \sigma_1 = \Delta \varepsilon_1 E(t - t_1) \]
\[ \Delta \sigma_0 = \Delta \varepsilon_0 E(t - t_0) \]
The Convolution Integral

\[ \sigma(t) = \Delta \varepsilon_0 E(t - t_0) + \Delta \varepsilon_1 E(t - t_1) + \Delta \varepsilon_2 E(t - t_2) + \ldots \]

Hence for continuously varying strain:

\[ \sigma(t) = \int_0^t E(t - \tau) d\varepsilon = \int_0^t E(t - \tau) \frac{d\varepsilon}{d\tau} d\tau \]
How are the viscoelastic properties of soft biological tissues determined?
Oscillations to determine viscoelastic properties

\[ \varepsilon = \hat{\varepsilon} \sin(\omega t) \]
Harmonic Loading

\[ y = A \sin(\omega t + \phi) \]
\[ x = A \cos(\omega t + \phi) \]

\[ z = x + iy = A e^{i \phi} \cos(\omega t + \phi) + i \sin(\omega t + \phi) \]
\[ = Ae^{i(\omega t + \phi)} = Ae^{i\phi} e^{i\omega t} = Be^{i\omega t} \]

A = amplitude = \( |B| \)
\[ \phi = \text{phase angle} = \tan^{-1}(\text{Im } B / \text{Re } B) \]
\[ \omega = \text{angular frequency} = 2\pi f \quad \text{(rad/s)} \]
Harmonic Stress and Strain History

\[ \varepsilon = \hat{\varepsilon} \cos(\omega t) \quad \varepsilon = \hat{\varepsilon} \cdot e^{i\omega t} \]

\[ \dot{\varepsilon} = i\omega \hat{\varepsilon} \cdot e^{i\omega t} \quad \Rightarrow \dot{\varepsilon} = i\omega \varepsilon \]

\[ \sigma = \hat{\sigma} \cos(\omega t) \quad \ldots \Rightarrow \dot{\sigma} = i\omega \sigma \]

Maxwell Model

\[ \varepsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta} \]

\[ \Rightarrow i\omega \varepsilon = \sigma \left( \frac{i\omega}{E_1} + \frac{1}{\eta} \right) \]

\[ \Rightarrow \sigma(i\omega) = i\omega \left( \frac{i\omega}{E_1} + \frac{1}{\eta} \right)^{-1} \varepsilon \]

\[ E(i\omega) \]
Complex modulus for the Maxwell Model

\[
\Rightarrow E(i\omega) = i\omega \frac{1}{i\omega E_1 + \frac{1}{\eta}} \times \left(\frac{i\omega}{E_1} - \frac{1}{\eta}\right)
\]

\[
\Rightarrow E(i\omega) = \frac{-\omega^2 - i\omega}{E_1 \frac{\omega^2}{E_1^2} + \frac{1}{\eta^2}}
\]

\[
\Rightarrow E(i\omega) = \left(\frac{\omega^2}{E_1} + \frac{i\omega}{\eta}\right) / \left(\frac{\omega^2}{E_1^2} + \frac{1}{\eta^2}\right)
\]

\[
E(i\omega) = E' + iE''
\]

"Stiffness"  
"Damping"
Dynamic modulus for the Maxwell Model

Dynamic modulus

$$|E(i\omega)| = \text{amplitude} = \sqrt{(\text{Re } E)^2 + (\text{Im } E)^2}$$

$$\Rightarrow |E(i\omega)| = \frac{\sqrt{\frac{\omega^4 + \omega^2}{E_1^2 + \frac{1}{\eta^2}}} \times \sqrt{E_1^2 \eta^2}}{\left(\frac{\omega^2}{E_1^2} + \frac{1}{\eta^2}\right)}$$

$$= \frac{\sqrt{\omega^2}}{\sqrt{E_1^2 + \frac{1}{\eta^2}}} \times \sqrt{E_1^2 \eta^2}$$

$$\Rightarrow |E(i\omega)| = \frac{\omega E_1 \eta}{\sqrt{E_1^2 + \eta^2 \omega^2}}$$
Internal friction for the Maxwell Model

\[ E(i\omega) = \left( \frac{\omega^2}{E_1} + \frac{i\omega}{\eta} \right) / \left( \frac{\omega^2}{E_1^2} + \frac{1}{\eta^2} \right) \]

**Internal friction**

\[ D = \tan \phi = \frac{\text{Im } E}{\text{Re } E} \Rightarrow \]

\[ \Rightarrow D = \frac{E_1}{\eta \omega} \]
Complex modulus for the Kelvin Model

\[ \sigma = E_1 \cdot \varepsilon + \eta \cdot \dot{\varepsilon} \]

\[ \dot{\varepsilon} = i \omega \varepsilon \Rightarrow \]

\[ \sigma(i \omega) = (E_1 + i \omega \cdot \eta) \varepsilon \]

\[ E(i \omega) = E_1 + i \omega \cdot \eta \]

Re \( E(i \omega) \)

"Stiffness"

Im \( E(i \omega) \)

"Damping"
Dynamic modulus and Internal Friction for the Kelvin Model

\[ E(i\omega) = E_1 + i\omega \cdot \eta \]

**Dynamic modulus**

\[ |E(i\omega)| = \text{amplitude} = \sqrt{(\text{Re} \ E)^2 + (\text{Im} \ E)^2} = \sqrt{E_1^2 + (\omega \cdot \eta)^2} \]

**Internal friction**

\[ D = \tan \phi = \frac{\text{Im} \ E}{\text{Re} \ E} = \frac{\omega \cdot \eta}{E_1} \]
Complex modulus for the SLS Model

Standard Linear Solid

\[ \varepsilon = i \omega \varepsilon \]
\[ \dot{\varepsilon} = i \omega \sigma \]

\[ \sigma = \frac{E_1 + i \omega \eta}{(E_1 + E_2) + i \omega \eta} \cdot E_2 \cdot \varepsilon \]

\[ E(i \omega) = \frac{E_1 + i \omega \eta}{(E_1 + E_2) + i \omega \eta} \cdot E_2 \Rightarrow \ldots \]

\[ E(i \omega) = \frac{E_1 E_2 (E_1 + E_2) + (\omega \eta)^2 E_2}{(E_1 + E_2)^2 + (\omega \eta)^2} + i \frac{\omega \eta E_2^2}{(E_1 + E_2)^2 + (\omega \eta)^2} \]
Dynamic Modulus and Internal Friction in the SLS Model

Dynamic modulus

\[ |E| = \sqrt{\frac{E_1^2 + (\omega \eta)^2}{(E_1 + E_2)^2 + (\omega \eta)^2}} E_2 \]

Internal friction

\[ D = \tan \phi = \frac{\omega \eta E_2}{E_1 (E_1 + E_2) + (\omega \eta)^2} \]

\[ \omega_{\text{peak}} = \sqrt{\frac{E_1 (E_1 + E_2)}{\eta}} \]
Hysteresis-Frequency Behavior
But soft biological tissues determined are not linear elastic...
Quasilinear Viscoelasticity

• Soft tissues exhibit several *viscoelastic* properties:
  – *hysteresis*
  – *stress relaxation*
  – *creep*
  – *strain-rate dependence*

• *Linear viscoelastic models* also display many of these properties

• However, soft tissue elasticity is *nonlinear*

• *Quasilinear viscoelasticity* combines the time history dependence of *linear viscoelasticity* with *nonlinear* elasticity
Quasilinear Viscoelasticity

• In soft tissue, the *elastic* response is *nonlinear*: \( \sigma = \sigma^{(e)}(e) \)
• However, the creep and relaxation functions can be *normalized* with reasonable accuracy, i.e. \( J_r(t) \) and \( E_r(t) \), are relatively independent of the initial strain or stress
• In *quasilinear viscoelasticity*, \( \sigma^{(e)}(e) \) can be nonlinear, but linear superposition still holds
• i.e., we *separate* the *time* and *load dependence* of the relaxation or creep response, e.g. \( E(e,t) = E_r(t) \sigma^{(e)}(e) \).
• Hence, the convolution integral for the stress is:

\[
\sigma(t) = \int_{0}^{t} E_r(t - \tau) \frac{\partial \sigma^{(e)}}{\partial t} d\tau = \int_{0}^{t} E_r(t - \tau) \frac{\partial \sigma^{(e)}(\varepsilon)}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial t} d\tau
\]
Viscoelasticity, Summary of Key Points

- The stress-strain relation is not unique, it depends on the load *history*.
- The elastic modulus depends on the load *history*.
- Simple spring-damper models give rise to one or more *first-order linear ODEs* which can be conveniently formulated and solved.
- *Creep, relaxation* and *hysteresis* are all properties of linear viscoelastic models.
Summary (continued)

- Creep solution can be normalized by the initial strain to give the reduced creep function \( J_r(t) \). \( J_r(0) = 1 \).
- Relaxation solution can be normalized by the initial stress to give the reduced relaxation function \( E_r(t) \). \( E_r(0) = 1 \).
- The strain response to an arbitrary stress history is obtained from \( J(t) \) by superposition

\[
\varepsilon(t) = \int_0^t J(t - \tau) d\sigma = \int_0^t J(t - \tau) \frac{d\sigma}{d\tau} d\tau
\]

- The stress response to an arbitrary strain history is obtained from the \( E(t) \) by superposition

\[
\sigma(t) = \int_0^t E(t - \tau) d\varepsilon = \int_0^t E(t - \tau) \frac{d\varepsilon}{d\tau} d\tau
\]
• The complex function $E(iw)$ depends on the frequency $w/2\pi$, and is called the \textit{complex modulus of elasticity}

• The magnitude of $E$ is the \textit{dynamic modulus of elasticity}

• The tangent of the phase angle $\frac{\text{Im}(E)}{\text{Re}(E)} = \tan\phi$ is called the \textit{internal friction} and represents the damping due to viscous elements

• Spring-damper models give rise to a \textit{finite number of peaks} in the Hysteresis-Frequency curve

• However, soft tissues have no discrete peaks. That is, soft tissue behave as though they have \textit{an infinite number of springs and dampers}

• \textit{Quasilinear viscoelasticity} combines the time history dependence of \textit{linear viscoelasticity} with \textit{nonlinear elasticity}
Change in schedule

TUESDAY 15/4 at 10-12 in E31
Injury mechanisms: Head
Svein Kliven

WEDNESDAY 16/4 at 13-15 in D34
Energy absorption
Peter Halldin