THE RELATION BETWEEN LOAD AND PENETRATION IN THE AXISYMMETRIC BOUSSINESQ PROBLEM FOR A PUNCH OF ARBITRARY PROFILE

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Abstract—A solution of the axisymmetric Boussinesq problem is derived from which are deduced simple formulae for the depth of penetration of the tip of a punch of arbitrary profile and for the total load which must be applied to the punch to achieve this penetration. Simple expressions are also derived for the distribution of pressure under the punch and for the shape of the deformed surface. The results are illustrated by the evaluation of the expressions for several simple punch shapes.

1. INTRODUCTION

The problem of determining (within the terms of the classical theory of elasticity) the distribution of stress within an elastic half space when it is deformed by the normal pressure against its boundary of a rigid punch seems to have been considered first by Boussinesq [1]. Using the methods of potential theory Boussinesq derived a solution of the problem corresponding to the case of penetration by a solid of revolution whose axis was normal to the original boundary of the half space, but the form of his solution did not lend itself to practical computations and partial numerical results based on his solution were derived only in the cases of a flat-ended cylindrical punch [2] and a conical punch [3].

After the publication of Boussinesq's solution several alternative solutions were derived, an excellent account of which is given in Galin's book [4]. From among these we may mention (because they are particularly relevant to the present paper) the solutions due to Harding and Sneddon [5] and Segedin [6].

The solution due to Harding and Sneddon uses the theory of Hankel transforms to express the axisymmetric solution of the equations of elastic equilibrium in terms of Hankel transforms of an arbitrary function and then to determine this arbitrary function by using a solution due to Titchmarsh [7] of the dual integral equations to which the mixed boundary problem can be reduced. The solution can be derived for the general case in which the equation of the punch in cylindrical coordinates with origin at the tip of the punch is \( z = w(\rho) \). The solutions appropriate to the cylindrical punch and the conical punch have been discussed in full in [8] and [9] respectively. In each case expressions are derived for \( D \), the total depth of penetration of the tip of the punch, and for \( P \), the total load on the punch, in terms of \( a \) the radius of the circle of contact between the punch and the elastic solid, but any attempt to express \( D \) and \( P \) in terms of an arbitrary function \( f \) is made difficult by the complicated form of the Titchmarsh solution of the dual integral equations.

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In the present paper the solution of the Boussinesq problem is again derived in the form of Hankel transforms of a function which is determined in terms of \( w(p) \) by the same pair of dual integral equations, but, instead of using Titchmarsh's solution, we make use of an elementary solution \[10\]. This enables us to derive the expressions

\[
D = \int_0^1 \frac{f'(x)dx}{\sqrt{(1-x^2)}} \quad P = 4\mu a(1-\eta)^{-1} \int_0^1 \frac{x^2f'(x)dx}{\sqrt{(1-x^2)}}
\]

where \( \mu \) and \( \eta \) are respectively the rigidity modulus and the Poisson ratio of the material of the half space and the function \( f \) is defined by the relation \( w(p) = f(p/a) \). Simple expressions (equations (3.4) and (5.1) below) are also derived for the distribution of pressure under the punch and for the shape of the deformed boundary of the half space.

In Section 6 the results corresponding to five special shapes of rigid punch are deduced from the general formulae, and in Section 7 the results corresponding to the case in which \( w(p) = \Sigma C_n p^n \) are deduced. The expressions obtained for \( D \) and \( P \) in the latter case are in agreement with those derived by Segedin \[6\].

2. SOLUTION OF THE BOUSSINESQ PROBLEM

The Boussinesq problem can be solved by means of a systematic use of Hankel transforms and the theory of dual integral equations.

The boundary conditions are

\[
\sigma_{\rho z}(\rho, 0) = 0, \quad u_\rho(\rho, 0) = D - f(p/a), \quad 0 \leq \rho \leq a \tag{2.1}
\]

and

\[
\sigma_{zz}(\rho, 0) = 0, \quad \rho > a, \tag{2.2}
\]

in which the function \( f \) is prescribed by the fact that, referred to the tip as origin, the punch has equation \( z = f(p/a) \) so that \( f(0) = 0; \) \( a \) is the radius of the circle of contact and \( D \) is a parameter (as yet unspecified) whose physical significance is that it is the depth to which the tip of the punch penetrates the elastic half space.

It is easily shown (see, e.g. \[11\] p. 452 et seq.) that the field specified by the equations \( u_\phi(\rho, z) = 0 \) and

\[
u_\rho(\rho, z) = -\frac{a}{2(1-\eta)} \mathcal{H}_1^0\left[\{(1-2\eta) - \xi z\}\xi^{-1}\psi(\xi a)e^{-\xi z}; \xi \to \rho\right], \tag{2.3}
\]

\[
u_z(\rho, z) = \frac{a}{2(1-\eta)} \mathcal{H}_0\left[\{(2(1-\eta) + \xi z\}\xi^{-1}\psi(\xi a)e^{-\xi z}; \xi \to \rho\right]. \tag{2.4}
\]

is a possible displacement field. In these equations \( \eta \) denotes the Poisson ratio of the material of the half space, the function \( \psi \) is arbitrary and we have used the notation

\[
\mathcal{H}_v\left[f(\xi, z); \xi \to \rho\right] = \int_0^\infty f(\xi, z)J_v(\xi \rho) d\xi
\]

to denote the Hankel transform of order \( v \) of the function \( f(\xi, z) \) with respect to the variable \( \xi \).

Inserting this form for the displacement vector \( (u_\rho, u_\phi, u_z) \) into the stress–strain relations we obtain the equations
The relation between load and penetration in the axisymmetric Boussinesq problem

\[ \sigma_{\rho z}(\rho, z) = -\frac{\mu az}{1-\eta} \mathcal{H}_0[\xi \psi(\xi a)e^{-\xi z}; \xi \to \rho] \]  
(2.5)

\[ \sigma_{zz}(\rho, z) = -\frac{\mu a}{1-\eta} \mathcal{H}_0[(1+\xi z)\psi(\xi a)e^{-\xi z}; \xi \to \rho] \]  
(2.6)

for the \(z\)-components of the stress tensor; the remaining component \(\sigma_{\theta z}\) vanishes identically. In these equations \(\mu\) represents the rigidity modulus (second Lamé constant) which in terms of Young's modulus \(E\) is given by the equation \(\mu = E/2(1+\eta)\).

From equations (2.4), (2.5), (2.6) we obtain the boundary values

\[ u_z(\rho, 0) = \mathcal{H}_0[\xi^{-1}\psi(\xi); \xi \to x] \]

\[ \sigma_{zz}(\rho, 0) = -\frac{\mu}{a(1-\eta)} \mathcal{H}_0[\psi(\xi); \xi \to x] \]

\[ \sigma_{\rho z}(\rho, 0) = 0, \]

where \(x = \rho/a\), so that the boundary conditions (2.1), (2.2) will be satisfied if we can find a function \(\psi(\xi)\) such that

\[ \mathcal{H}_0[\xi^{-1}\psi(\xi); \xi \to x] = D - f(x), \quad 0 \leq x < 1, \]  
(2.7)

\[ \mathcal{H}_0[\psi(\xi); \xi \to x] = 0, \quad x > 1. \]  
(2.8)

These dual integral equations can be solved by an elementary method [10]. If we represent \(\psi(\xi)\) by a formula of the type

\[ \psi(\xi) = \int_0^1 \chi(t) \cos(\xi t) \, dt \]  
(2.9)

we find that equation (2.8) is automatically satisfied and that equation (2.7) is equivalent to the Abel integral equation

\[ \int_0^x \frac{\chi(t) \, dt}{\sqrt{x^2 - t^2}} = D - f(x), \quad 0 \leq x \leq 1 \]

for the determination of the function \(\chi(t)\). The solution of this equation is known to be

\[ \chi(t) = \frac{2D}{\pi} - \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{xf(x) \, dx}{\sqrt{(t^2 - x^2)}}. \]  
(2.10)

The solution of the Boussinesq problem is therefore given in terms of \(f(x)\) and \(b\) by equations (2.3), (2.4), (2.9) and (2.10). The constant \(D\) occurring in this solution is still arbitrary. We now proceed to determine it by making an additional physical assumption.

3. FORMULA FOR THE DEPTH OF PENETRATION OF THE PUNCH

We now determine \(D\), the depth of penetration of the tip of the punch, from the physical condition that if the punch has a smooth profile the normal component of stress \(\sigma_{zz}\) must remain finite round the circle of contact between the punch and the elastic solid.†

Since from equations (2.6) and (2.9)

† It should be noted that this condition is not satisfied in the case of a flat-ended cylindrical punch because the smoothness condition is violated on the circular base of the punch, \(a\) being equal to the radius of the cylinder.
\[ \sigma_{zz}(\rho, 0) = -\frac{\mu}{a(1-\eta)} \mathcal{H}_0 \left[ \int_0^1 \chi(t) \cos(\zeta t) \, dt; \, \zeta \to x \right] \]  

(3.1)

and since for any arbitrary function \( g(\zeta) \) we have the relation

\[ \frac{1}{x} \frac{d}{dx} x \mathcal{H}_1 \left[ \zeta^{-1} g(\zeta); \, x \right] = \mathcal{H}_0 \left[ g(\zeta); \, x \right] \]

we find that

\[ \sigma_{zz}(\rho, 0) = -\frac{\mu}{a x(1-\eta)} \frac{d}{dx} x \mathcal{H}_1 \left[ \zeta^{-1} \int_0^1 \chi(t) \cos(\zeta t) \, dt; \, \zeta \to x \right]. \]  

(3.2)

Now if we interchange the order of the integrations we find that

\[ \mathcal{H}_1 \left[ \zeta^{-1} \int_0^1 \chi(t) \cos(\zeta t) \, dt; \, \zeta \to x \right] = \int_0^1 \chi(t) \, dt \int_0^\infty J_1(\xi x) \cos(\zeta t) \, d\zeta \]

\[ = \frac{1}{x} \int_0^x \chi(t) \, dt + \frac{1}{x} \int_x^1 \chi(t) \left( 1 - \frac{t}{\sqrt{(t^2 - x^2)}} \right) \, dt \]

\[ = \frac{1}{x} \int_0^1 \chi(t) \, dt - \frac{1}{x} \int_x^1 \frac{t \chi(t) \, dt}{\sqrt{(t^2 - x^2)}}. \]

If we substitute this expression into the right hand side of equation (3.2) we have the formula

\[ \sigma_{zz}(\rho, 0) = -\frac{\mu}{a x(1-\eta)} \frac{d}{dx} \int_0^1 \frac{t \chi(t) \, dt}{x \sqrt{(t^2 - x^2)}}, \quad (\rho = xa), \]  

(3.3)

which can be written in the alternative form

\[ \sigma_{zz}(\rho, 0) = -\frac{\mu}{a(1-\eta)} \left\{ \frac{\chi(1)}{\sqrt{(1-x^2)}} - \int_x^1 \frac{\chi(t) \, dt}{\sqrt{(t^2 - x^2)}} \right\}. \]  

(3.4)

If we now take \( \rho = a(1+\delta) \) where \( \delta \) is small and positive and change the variable of the integral on the right from \( t \) to \( u = 1 - t \) we find that to the first order in

\[ \sigma_{zz}(a - a\delta, 0) = -\frac{\mu}{1-\eta} \left\{ \frac{\chi(1)}{\sqrt{(2\delta)}} - \int_0^{\delta} \frac{x(1-u) \, du}{\sqrt{2(\delta - u)}} \right\}. \]

Now if \( \chi(t) \) is differentiable in the neighbourhood of \( t = 1 \) we see that this result can be written as

\[ \sigma_{zz}(a - a\delta, 0) = -\frac{\mu}{1-\eta} \frac{\chi(1)}{\sqrt{(2\delta)}} + O(\delta). \]

Hence if \( \sigma_{zz}(a - a\delta, 0) \) tends to a finite limit as \( \delta \to 0^+ \) we must have

\[ \chi(1) = 0 \]  

(3.5)

We can use this criterion to determine the constant \( D \) by writing equation (2.10) in the form

\[ \frac{1}{2} \pi \chi(t) = D - t \int_0^t \frac{f'(x) \, dx}{\sqrt{(t^2 - x^2)}} \]  

(3.6)
and then putting $t=1$ to obtain the relation

$$D = \int_0^1 \frac{f'(x) \, dx}{\sqrt{1-x^2}}$$  \hspace{1cm} (3.7)$$

for the total depth of penetration of the tip of the punch.

4. FORMULA FOR THE TOTAL LOAD ON THE PUNCH

The total load $P$ on the punch required to produce the above penetration is given by the equation

$$P = -2\pi \int_0^a \rho \sigma_{rs}(\rho, 0) \, d\rho$$

$$= 2\pi \mu a \int_0^1 \xi \psi(\xi) \, d\xi \int_0^a \rho J_0(\rho \xi/a) \, d\rho$$

from which it follows that

$$P = \frac{2\pi \mu a}{1-\eta} \int_0^\infty \psi(\xi) J_1(\xi) \, d\xi .$$

If we use the representation (2.9) and the integral

$$\int_0^\infty \cos(\xi t) J_1(\xi) \, d\xi = 1, \quad 0 < t < 1,$$

we find that we can write this relation in the form

$$P = \frac{2\pi \mu a}{1-\eta} \int_0^1 \chi(t) \, dt . \hspace{1cm} (4.1)$$

Now if we combine equations (2.10) and (2.7) we find that $\chi(t)$ is given by the equation

$$\chi(t) = \frac{2}{\pi} \left\{ \int_0^1 \frac{f'(x) \, dx}{\sqrt{1-x^2}} - t \int_0^t \frac{f'(x) \, dx}{\sqrt{1-x^2}} \right\} . \hspace{1cm} (4.2)$$

If we substitute this expression into equation (4.1) and carry out the integration we obtain the formula

$$P = \frac{4\mu a}{1-\eta} \int_0^1 \frac{x^2 f'(x) \, dx}{\sqrt{1-x^2}} \hspace{1cm} (4.3)$$

by means of which we can calculate the total load $P$ necessary to produce the depth of penetration (3.8) by a rigid punch whose profile is defined by the function $f$.

5. SHAPE OF THE DEFORMED SURFACE

It is also of interest to know the form of $u_*(\rho, 0)$ when $\rho > a$. We derive this from the relation

$$u_*(\rho, 0) = \mathcal{H} \left[ \xi^{-1} \int_0^1 \chi(t) \cos(\xi t) \, dt ; \quad \xi \to x \right] ,$$
and the integral
\[ \int_0^\infty J_0(\xi x) \cos(\xi t) \, d\xi = (x^2 - t^2)^{-1/2} H(x - t). \]
This leads to the relation
\[ u_x(ax, 0) = \int_0^1 \frac{\chi(t) \, dt}{\sqrt{(x^2 - t^2)}}, \quad x > 1. \]  
(5.1)

If we substitute the expression (4.2) for \( \chi(t) \) into this equation and use the fact that
\[ \int_0^1 f'(y) \, dy \int_0^1 \frac{t \, dt}{y \sqrt{[(x^2 - t^2)(t^2 - y^2)]}} = \int_0^1 f'(y) \left\{ \frac{1}{2} \pi \sin^{-1} \frac{\sqrt{(x^2 - 1)}}{\sqrt{(x^2 - y^2)}} \right\} \, dy \]
we find that
\[ u_x(ax, 0) = \frac{2D}{\pi} \sin^{-1}(1/x) + \frac{2}{\pi} \sqrt{(x^2 - 1)} \int_0^1 \frac{y f(y) \, dy}{(x^2 - y^2)\sqrt{(1 - y^2)}}, \]  
(5.2)

where \( D \) is the depth of penetration of the tip given by the equation (3.7).
In any practical problem it may be easier to calculate the normal component of the surface displacement from the pair of formulae (5.1) and (4.2) than from the single formula (5.2).

6. RESULTS FOR SPECIAL SHAPES OF PUNCH

We shall now consider some special cases of the application of these formulae.

(a) Flat-ended cylindrical punch
We begin by considering the case in which the half space \( z \geq 0 \) is deformed by the normal penetration of the boundary by a flat-ended rigid cylinder of radius \( a \). We suppose that the punch penetrates a distance \( D \). Since the profile of the punch is not smooth at \( \rho = a \) we do not have the condition that \( \sigma_{xx}(a, 0) \) is finite and must regard \( D \) as one of the data of the problem. In equation (2.10) we take \( f(x) = 0 \) and obtain the simple equation \( \chi(t) = 2D/\pi \). Substituting this expression into equation (4.1) we find that the total load required to produce a penetration \( D \) is given by the equation
\[ P = \frac{4\mu a D}{1 - \eta}. \]  
(6.1)

Similarly from equation (3.4) we find that the distribution of pressure under the punch is given by the equation
\[ \sigma_{xx}(\rho, 0) = -\frac{2\mu D}{\pi(1 - \eta)} (a^2 - \rho^2)^{-1/2}, \quad 0 < \rho < a, \]  
(6.2)

and from equation (5.1) we find that the shape of the deformed boundary is given by the relation
\[ u_x(\rho, 0) = \frac{2D}{\pi} \sin^{-1}(a/\rho), \quad \rho > a. \]  
(6.3)
These expressions are in agreement with known results; see, for example, [8] or (pp. 460–461 of [11]).

(b) Conical punch

For normal penetration by a cone of semivertical angle \( \alpha \) we may take \( f(x) = \varepsilon x \) where \( \varepsilon = a \tan \alpha \).

From equation (3.7) we find that the vertex of the cone penetrates to a distance \( D = \frac{1}{2} \pi \varepsilon = \frac{1}{2} \pi a \tan \alpha \) and from equation (4.3) that the total load necessary to effect this penetration is given by the equation

\[
P = \frac{\pi \mu a^2}{1-\eta} \tan \alpha ,
\]

which can be written in the alternative form

\[
P = \frac{4 \mu \cot \alpha}{\pi (1-\eta)} D^2 .
\]

From equation (2.10) we find that in this case

\[
\chi(t) = \frac{2D}{\pi} (1 - t).
\]

Substituting this expression into equation (5.1) we find that the deformed shape of the free surface is given by the relation

\[
u_x(\rho, 0) = \frac{2D}{\pi a} \left\{ a \arcsin(\alpha/\rho) - \rho + \sqrt{(\rho^2 - a^2)} \right\}, \quad \rho > a ,
\]

and substituting it into equation (3.4) we find that the distribution of pressure under the punch is given by the relation

\[
\sigma_{xx}(\rho, 0) = -\frac{2\mu D}{(1-\eta)\pi a} \cosh^{-1}(\alpha/\rho) , \quad 0 < \rho < a .
\]

The equations (6.5), (6.6), (6.7) are in agreement with the results obtained by other methods, see, for instance, [3], [9] or (pp. 462–466 of [11]).

(c) Punch in the form of a paraboloid of revolution

If the punch is a paraboloid of revolution with equation \( \rho^2 = 4kz \) we may take \( f(x) = \varepsilon x^2 \) where \( \varepsilon = a^2/4k \). If we substitute this expression into equation (3.7) we find that the relation between \( D \) and \( a \) is

\[
a^2 = 2kD ,
\]

so that the depth of penetration of the paraboloid is twice the distance that the circle of contact lies below the original boundary of the half space.

In a similar way equation (4.3) leads to the expression

\[
P = \frac{16\mu \varepsilon}{3(1-\eta)}
\]

for the total load. In terms of the total penetration \( D \) this may be written in the form

\[
P = \frac{8\mu}{3(1-\eta)} (2kD^3)^{\frac{1}{4}}
\]
From equation (2.10) we find that
\[ \chi(t) = \frac{2D}{\pi} (1 - t^2), \]
so that we find from equation (3.3) that the distribution of stress under the punch is given by the equation
\[ \sigma_{xx}(\rho, 0) = -\frac{4\mu D}{\pi a^2 (1 - \eta)} \sqrt{(a^2 - \rho^2)}, \quad 0 \leq \rho < a, \quad (6.10) \]
and from equation (5.1) that the shape of the free surface is given by the equation
\[ u_z(\rho, 0) = \frac{D}{\pi} \left\{ (2 - \rho^2/a^2) \sin^{-1}(a/\rho) + \frac{D}{a} \sqrt{(1 - a^2 \rho^2)} \right\}, \quad \rho > a. \quad (6.11) \]

(d) **Spherical punch**

The case of a spherical indenter is of interest for practical applications. We assume that the sphere is of radius \( R \) and (as in the previous cases) that it fits the elastic solid over a circle of radius \( a \). Hence we may take
\[ f(x) = R - \sqrt{(R^2 - a^2 x^2)} \]
in equation (2.1).

For this function we have
\[ \int_0^t f'(x) \frac{dx}{\sqrt{(t^2 - x^2)}} = a^2 \int_0^t \frac{x dx}{\sqrt{(t^2 - x^2)(t^2 - x^2)}}. \]
If we change the variable in the integral on the right from \( x \) to \( w = t^{-1} \sqrt{(t^2 + x^2)} \) we find that
\[ t \int_0^t f'(x) \frac{dx}{\sqrt{(t^2 - x^2)}} = \frac{1}{2} at \log \frac{R + at}{R - at}. \quad (6.12) \]
Putting \( t = 1 \) in this result and then substituting into equation (3.7) we find that the relation between \( a \) and \( D \) is given by the equation
\[ D = \frac{1}{2} a \log \frac{R + a}{R - a}. \quad (6.13) \]
Substituting from equation (6.12) into equation (3.6) we find that in this case
\[ \chi(t) = \frac{2D}{\pi} - \frac{at}{\pi} \log \frac{R + at}{R - at}. \quad (6.14) \]
Using the fact that
\[ \int_0^1 t \log \frac{R + at}{R - at} dt = \frac{1}{2} \left( 1 - \frac{R^2}{a^2} \right) \log \frac{R + a}{R - a} + \frac{R}{a}, \]
we see from equation (4.1) that the total load \( P \) on the sphere necessary to produce the penetration (6.13) is given by the equation
\[ P = \frac{\mu}{1 - \eta} \left\{ (a^2 + R^2) \log \frac{R + a}{R - a} - aR \right\}. \quad (6.15) \]
(e) Punch in the shape of an ellipsoid of revolution

If the shape of the indenting punch is an ellipsoid of revolution with semi-axes $\alpha$, $\beta$, $\beta$ (the axis of length $2\alpha$ coinciding with the z-axis) then we may take

$$f(x) = \alpha\{1 + \sqrt{(1 + \alpha^2 x^2/\beta^2)}\}.$$ 

Hence it follows from equation (3.7) that

$$D = \frac{\alpha a^2}{\beta^2} \int_0^1 \frac{x \, dx}{\sqrt{[(1-x^2)(1-a^2 x^2/\beta^2)]}}.$$ 

The integration is elementary and we find that

$$D = \frac{\alpha a}{2y} \log \frac{\beta + \alpha}{\beta - \alpha}. \quad (6.16)$$ 

Similarly from equation (4.3) we find that

$$P = \frac{\mu \alpha}{(1-\eta)\beta} \left[(\alpha^2 + \beta^2)\log \frac{\beta + \alpha}{\beta - \alpha} - 2\beta \alpha\right]. \quad (6.17)$$

7. DERIVATION OF SEGEDIN’S FORMULAE

Segedin [6] has considered the case in which the profile of the punch has equation

$$z = \sum_{n=1}^{\infty} c_n \rho^n \quad (7.1)$$

in which case we have

$$f(x) = \sum_{n=1}^{\infty} c_n \alpha^n x^n.$$ 

If we substitute this expression into equation (3.7) we find that the appropriate formula for the total penetration of the depth of the punch is

$$D = \sqrt{\pi} \sum_{n=1}^{\infty} \frac{\Gamma(\frac{1}{2} n + 1)}{\Gamma(\frac{1}{2} n + \frac{1}{2})} c_n a^n, \quad (7.2)$$

and if we substitute it into equation (4.3) we find that the total load $P$ is given by the formula

$$P = \frac{2\sqrt{\pi} \mu a}{1-\eta} \sum_{n=1}^{\infty} \frac{n \Gamma(\frac{1}{2} n + 1)}{\Gamma(\frac{1}{2} n + \frac{1}{2})} c_n a^n, \quad (7.3)$$

in agreement with Segedin’s results.

Similarly from equation (2.10) we obtain the expression

$$\chi(p) = \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{\Gamma(\frac{1}{2} n + 1)}{\Gamma(\frac{1}{2} n + \frac{1}{2})} c_n a^n (1-p). \quad (7.4)$$

Hence from equation (3.4) we find that the formula for the distribution of pressure under the punch is

$$\sigma_{xx}(\rho, 0) = -\frac{2\mu}{a(1-\eta)} \sum_{n=1}^{\infty} \frac{n \Gamma(\frac{1}{2} n + 1)}{\Gamma(\frac{1}{2} n + \frac{1}{2})} c_n a^n i_n(x), \quad (x = \rho/a, \quad 0 \leq x < 1), \quad (7.5)$$
where the function $i_n(x)$ is defined by the equation
\[
i_n(x) = \frac{\int_0^1 r^{n-1}dt}{x \sqrt{(t^2-x^2)}}, \quad (n \geqslant 1, \ 0 \leqslant x < 1), \tag{7.6}
\]
and from equation (5.1) that the shape of the deformed surface is given by the relation
\[
u_z(\rho, 0) = \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{1}{2}n + 1\right)}{\Gamma\left(\frac{3}{2}n + \frac{1}{2}\right)} c_n a^n \left[\sin^{-1}(1/x) - j_n(x)\right], \tag{7.7}
\]
with $x = \rho/a > 1$ and
\[
j_n(x) = \frac{\int_0^1 r^n dt}{\sqrt{(x^2 - t^2)}}, \quad (n \geqslant 1, \ x > 1). \tag{7.8}
\]
The integrals $i_n(x)$ can be determined by the recursion relation
\[
(n+1)i_n(x) = \sqrt{1-x^2} + (n-2)x^2i_{n-2}(x), \quad (n \geqslant 3)
\]
with
\[
i_1(x) = \cosh^{-1}(1/x), \quad i_2(x) = \sqrt{1-x^2}, \quad (0 \leqslant x < 1).
\]
Similarly, the integrals $j_n(x)$ can be determined by the recursion relation
\[
nj_n(x) = (n-1)x^2j_{n-2}(x) - \sqrt{x^2-1}, \quad (n \geqslant 2),
\]
with
\[
j_0(x) = \sin^{-1}(1/x), \quad j_1(x) = x - \sqrt{x^2-1}, \quad (x > 1).
\]

REFERENCES


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The relation between load and penetration in the axisymmetric Boussinesq problem


Sommario—Si deriva una soluzione del problema assisimmetrico del Boussinesq dalla quale si deducono semplici formule per la profondità di penetrazione della punta di un punzone di profilo arbitrario e per il carico totale che deve venire applicato al punzone per ottenere detta penetrazione. Si derivano inoltre semplici espressioni per la distribuzione della pressione sotto il punzone e per il profilo della superficie deformata. I risultati sono illustrati con la valutazione delle espressioni per vari profili semplici di punzone.

Абстракт—Дано решение аксиосимметричной проблемы Буссинеск, из которого выведены простые формулы для глубины погружения конца пробойника произвольного профиля, и для полной нагрузки, которая должна быть приложена к пробойнику, чтобы дать эту глубину погружения. Также выведены простые выражения для распределения давления под пробойником и для формы деформированной поверхности. Результаты иллюстрированы вычислениями таких выражений для нескольких простых форм пробойников.