


time ago. We complete this review (section on ‘Applications’) with a brief description of some recent applications (mostly to nanoindentation) and an example of controlled damage by nanoindentation on initially defect free silicon wafer samples.

**Indentation fracture**

A great deal of study has been made on the indentation fracture of glasses and ceramics, and whereas we might expect differences in detail between the phenomena observed in these materials and those observed in brittle rock, there are nevertheless fairly close similarities between them. In this section, we describe the main features likely to be common to indentation induced fracture patterns and attempt a quantitative description of them. There is a vast literature on the indentation fracture of ceramics and glasses (see the references at the end of this review for a selection).

The objectives for the first parts of this section will be to:

(i) characterise the geometry and extent of indentation induced cracking in terms of contact stress field

(ii) derive expressions for the scale of microfracturing in terms of the indentor load from which the role of material properties (e.g. hardness \( H \), toughness \( \Gamma \) and stiffness \( E \)) and extraneous variables (e.g. indentor geometry, environment, etc.) may be inferred.

As far as the loading stress field is concerned, a crude distinction can be made between the types of indentor, i.e. ‘blunt’ or ‘sharp’. For a point indentation, typified by a conical indentor, qualitative results are that:

(i) blunt indentors: dominant tension develops immediately below penetration point. Hertzian contact cracks observed when spheres impact glass

(ii) sharp indentors: dominant tension develops immediately below penetration point. Hertzian contact cracks observed when spheres impact glass

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This distinction can be seen most clearly by comparing the Hertzian and Boussinesq stress fields created by spherical and point indentors respectively (see Appendixes 1 and 2).
Contact mechanics

1 Contact mechanics

The brief description given above emphasises the influence of the tensile stress field since this is most likely to directly influence the crack patterns. Note, however, that other stress components, e.g. shear and hydrostatic, exist in the indentation field, and these control the operation of irreversible deformation modes (plastic flow, densification and crushing) responsible for the hardness impression and residual stress fields that may occur during indentation or upon indentor removal.

Evolution of crack patterns

The three stages in the evolution of fracture patterns are crack nucleation, formation and propagation. The propagation stage is very important to chip formation and, in particular, lateral crack system to be described later. Nevertheless, the damage created during indentation must affect the effective permeability of the rock during subsequent indentations and possibly has an influence on indexing.

Crack nucleation

A typical brittle solid contains a profusion of pre-existing flaws, and the deformation process during indentation may itself create crack nuclei by micro-mechanical slip events. The macroscopic inhomogeneous stress field induced by the indentor interacts with these so that pre-existing near surface flaws dominate crack initiation in the case of blunt indentors. This leads to Hertzian conical cracks for blunt conical indentors and to prismoidal shaped cracks for blunt chisel shaped indentors. These cracks are indicated in cross-section in Fig. 1 by the letters C and P.

For sharp indentors, deformation induced flaws below the surface dominate since high tensile stresses are created there by the inhomogeneous (sharp) indentor stress field.

Note that more subtle explanations may be relevant if the micromechanics of non-elastic deformation beneath the indentor are looked at in more detail.

Crack formation

The dominant flaws begin to extend into the solid (stably) as the indentor load is increased. This can be described in terms of an energy barrier to full crack propagation due to the stress cutoff of non-zero contact area. The depth to which a crack must grow before overcoming this energy barrier is usually small compared to the dimension of the contact area. This leads to the crack propagation stage.

Crack propagation

Once the formation energy is exceeded for a given crack, rapid crack propagation follows until at a depth much greater than the contact dimensions, the tensile driving force falls below that necessary to maintain growth. Thus, a stable crack is produced, and the near contact stress field is less important.

Cases that have been studied in glasses are as follows.

Hertzian cone cracks

These have been observed as initiating from a surface flaw by elastic contact with a normally loaded sphere. They are shown in Fig. 1 indicated by the letter C.

'Median' half penny cracks

These are initiated from a deformation induced flaw by the tip of a sharp cone or pyramid indentor. There could be several such cracks depending upon existing inhomogeneities in the material and the symmetry of the indentor and hence the stress field induced by it. Figures 2 and 3 show a side view of such cracks produced by wedge and cone shaped indentors respectively. The crack will propagate under the influence of the tensile stress induced by the indentor.

Lateral vents

The picture that is usually given in the ceramics and glass literature is that, on unloading, the reversal of the indentation load reduces the tensile opening stress on the median and cone cracks, and they tend to close but rarely heal. However, there may still exist a residual stress due to the mismatch between the irreversible deformed material and the surrounding elastic matrix, which may still drive median cracks and other radial cracks that may have been initiated in the deformation zone beneath the indentor. In addition, these residual stresses create conditions for the initiation and propagation of the lateral crack system shown in Fig. 4. In extreme loading cases, these lateral cracks are responsible for chip removal as they propagate to the surface or intersect with other crack systems, such as the Hertzian cone cracks, to cause craters.

Qualitative relations between scale of cracking and applied load

For advanced stages of loading, the following geometrical approximations are made:

(i) for 'point load indentors' (i.e. conical indentor), cone and median cracks are considered to be 'penny-like' (expanding on an ever increasing circular front)
(ii) for 'line force indentors' (i.e. wedge shaped indentors), the median vents and prism shaped cracks are viewed as through the surface planar cracks, as shown in Fig. 2.

Median vents and Hertzian cone cracks

To obtain the basic relationships between the scale of cracking and the applied load, the balance between the mechanical energy $U_M$ and the surface energy $U_S$ is considered for virtual displacement $\delta c$ in the crack system. Consider the situations shown in Figs. 2 and 3. For these situations, one gets for the change in the surface energy $\delta U_S$

$$\delta U_S \propto \Gamma(L\delta c) \quad \text{for line contact}$$

$$\delta U_S \propto \Gamma(c\delta c) \quad \text{for point contact}$$

(1)

where $\Gamma$ is the fracture surface energy (energy required to create a unit area of a new crack surface), and $c$ is a characteristic crack length, and expression (1) is shown...
as proportional since similarity relations are only anticipated by this argument. For the median crack system shown in Fig. 3, we might, of course, prefer \( \frac{P}{d} \) instead of \( \frac{c}{d} \) in expression (1). To estimate the mechanical energy, consider the applied load \( P \) with \( P = P_L L \) if acting over a line such as shown in Fig. 2. The stress level of the indentation field is estimated as the load divided by a characteristic area. For the stress at a distance \( c \) from the load point, the characteristic area is assumed to be the area of the surface everywhere at distance \( c \) from the contact point supporting this load. Hence, the stress level \( \sigma \) is given by

\[
\frac{\sigma}{2} \propto \frac{P}{(Lc)} \quad \text{for line contact}
\]

\[
\frac{\sigma}{2} \propto \frac{P}{c^2} \quad \text{for point contact}
\]

The strain energy density \( w \) (for a linear elastic material with Young’s modulus \( E \)) varies as \( \frac{\sigma^2}{(2E)} \), so neglecting the influence of Poisson’s ratio, this gives

\[
w \propto \frac{P^2}{(Lc^2E)} \quad \text{for line contact}
\]

\[
w \propto \frac{P^2}{(c^4E)} \quad \text{for point contact}
\]

The volume of the stressed material (\( \delta V \), say) associated with the crack extension is that traced out by the characteristic support area, which is proportional to \( Lc \delta c \) for line contact and to \( c^2 \delta c \) for point contact. Then, the change in strain energy is \( \delta U_M \propto -w \delta V \), giving

\[
\delta U_M \propto -\frac{P^2 \delta c}{(LcE)} \quad \text{for line loads}
\]

\[
\delta U_M \propto -\frac{P^2 \delta c}{(c^4E)} \quad \text{for point loads}
\]

The negative sign indicates that the mechanical energy diminishes as the crack extends.

Now, the energy balance for crack equilibrium requires that the total energy change of the system is zero. Hence

\[
\delta U_M + \delta U_S = 0
\]

and from equations (1) and (4), we get

\[
P^2 / (c^4E) \propto \frac{1}{(c^4E)}
\]

for point loads (6b)

recalling that \( P = P_L L \) for line loads. From relation (6a) and (6b), the equilibrium length of median and cone cracks can be determined in terms of the applied load once the surface energy (or toughness) \( \Gamma \) of the material is known as well as its elastic modulus.

An argument similar to that above has been given by Roesler\(^5,6\) for cone cracks and by Lawn and co-workers\(^7\) in a number of papers. To determine the proportionality constants in expression (6a) and (6b), a more complete analysis is necessary. Such an analysis must take into account the inhomogeneous stress field beneath the indentor and use this to predict the direction in which cracks will propagate and then calculate the stress intensity factors and energy release rates associated with the fully developed crack. This approach will be outlined more fully in the next section.

We close this section by deriving a scaling law for the lateral crack system following Swain and Lawn\(^10\).

**Lateral crack system**

Some idea of the crater volume produced by fully developed lateral vents can be obtained if it is assumed that the residual driving forces produced by the inelastic zone beneath the indentor (Fig. 4) are of sufficient intensity that the lateral cracks will intersect the free surface. A reasonable hypothesis is then that the size of...
4 Lateral crack system: residual force $P_r$ exerted by radially expanded plastic zone determines crack driving force; broken lines indicate median radial crack system

the resultant chip should scale with that of the hardness impression, since the inelastic deformation associated with this hardness impression constitutes the source of the residual stress field. At least this is the picture currently accepted in the ceramics literature. It seems a plausible starting point for a scaling law for crack formation in rock. For geometrically similar indentations, the characteristic dimension of the residual impression (Fig. 4) can be related to the peak load $P^*$ through the standard hardness (mean contact pressure) relations

$$H = \frac{P^*}{(2La)} \text{ for line loads}$$
$$H = \frac{P^*}{(\pi a^2)} \text{ for point loads} \quad (7)$$

where $a$ is a factor depending on the indentor geometry ($a=1$ for axially symmetric indentors). The characteristic dimension $c'$ of the lateral crack, shown in Fig. 4, can thus be written as

$$c' = \lambda_L a = \frac{P^*}{2HL} = \lambda_L \frac{P_1^*}{2H} \text{ for line loads} \quad (8a)$$

$$c' = \lambda_P a = \frac{P^*}{2\pi H} \left( \frac{1}{2} \right) \text{ for point loads} \quad (8b)$$

The $\lambda$ terms are scaling factors, which will probably depend upon the indentor geometry and perhaps also the material properties. Note that hardness is the key parameter in this case since this is assumed to control the residual tensile stress field. The fracture toughness is important in specifying the resistance to lateral crack formation when the applied loading is low, but here, we have assumed that the applied load is sufficient to drive the lateral cracks to the free surface.

Most experiments on indentation fracture have been carried out on glasses and ceramics since glasses in particular allow observations on fracture patterns to be relatively easily. However, some tests of how the above expressions may apply to rock have been given by Swain and Lawn.10 If the proportionality constant in expression (6) is assumed to depend only on the indentor geometry, then for invariant indentor geometry, equation (6b) should give for wedge indentation

$$\frac{\Gamma_2}{\Gamma_1} = \frac{E_2}{E_1} \left( \frac{P_1^*}{P_{1L}} \right)^2 \frac{c_1}{c_2} \quad (9)$$

Comparing these results for granite and glass (they take for their soda lime glass $\Gamma=3.9 \text{ J m}^{-2}$, $E=73 \text{ GPa}$ and $v=0.25$, with Young’s modulus $E=34 \text{ GPa}$ for westerly granite), they predict from equation (9) a value for the fracturing energy of the granite of $\sim 30 \text{ J m}^{-2}$, which is of the same order of magnitude as the result ($\Gamma=100 \text{ J m}^{-2}$) given by Hoagland et al.11 by means of a fracture mechanics test. Taking into account the variability of rock specimens, these results do give a degree of plausibility to the above theoretical results. Swain and Lawn10 also made the observation that the trend away from median cracking towards cone cracking in moving from sharp to blunt indentors is less evident in granite than in glass, and they attribute this to a difference in crack initiation, in particular to the presence of pre-existing internal flaws in the granite.

For the lateral crack system, expression (8) can be used in a similar way to that followed in deriving equation (9) if it is assumed that $\lambda_L$ and $\lambda_P$ depend only on the indentor geometry. With this assumption and invariant indentor geometry acting on two different materials, expression (8b) gives

$$\frac{H_2}{H_1} = \frac{P_2^*}{P_1^*} \left( \frac{c_1}{c_2} \right)^2 \text{ for point loads} \quad (10)$$

Swain and Lawn10 claim to use this expression together with data resulting from their experiments to compute $H$ (granite)=$7.3 \text{ GPa}$ [they take $H$ (glass)=$5.5 \text{ GPa}$]. They claim that independent hardness measurements using a Vickers diamond pyramid (with $P^*=100 \text{ N}$) gave $H$(granite)=$4.9 \pm 0.3 \text{ GPa}$. They do not give extensive data for their experiments on rock; most of the data in their paper refer to glass. Nevertheless, the indentation induced crack patterns following the tensile stress created by the inhomogeneous indentation stress fields seem highly plausible, and the influence of hardness on the lateral crack system is worth consideration.

Fracture mechanics analysis of indentation fracture

Although the functional dependence between crack size and indentor load described by equation (6a) and (6b) seems to be verified by the available experimental data, the description remains incomplete. To account for the influence of contact geometry on the ensuing fracture process, a more detailed analysis is necessary. In principle, this can be carried out by performing the following sequence of steps:

(i) calculate the inhomogeneous stress field produced by the indentor
(ii) postulate crack initiation of those positions, where the maximum tensile stress is produced provided it is large enough
(iii) follow the crack propagation process using the indentation induced stress field as a driving force for the fracture.

As might be expected, there are a number of complications involved in accomplishing the above steps. For the first step, some assumptions must be made about the response of the indented material to the applied load. If this response is assumed to be purely elastic, then a stress field such as that shown in Appendix 1 for Hertzian contact is appropriate for blunt indentors. For sharp indentors, an approximation to the stress field far from the indentor tip can be obtained using the Boussinesq stress field (shown in Appendix 2) evaluated
the indentor geometry for blunt indentors. Roesler gives a
equilibrium position of the crack, i.e. the release rate
n
The constant analysis that Roesler gave, for fully developed cone cracks, the
which they spread can be controlled. Experiments by
lot of attention since they are stable and the speed at
Conical indentation fractures in glasses have received a
account.

(i) the flaw first propagates around the circle of contact and subsequently grows downward into
the material as a surface ring; this flaw is assumed to have initiated from a perpendicular
flaw of length \( e_t \) located at position \( (x_c,0) \) in the specimen surface (Fig. 5).
(ii) the downward propagating surface crack is approximated by a plane edge crack; this is valid
provided that \( c << a \) because the crack curvature is neglected
(iii) the crack path at any instant is assumed to follow the direction of the principal stress trajectories passing through the crack tip.

Having made the above assumptions, Lawn et al.\(^7\) define the stress intensity factor as
\[
K = 2 \left( \frac{\varepsilon}{\eta} \right)^{1/2} \int_0^\infty \frac{\sigma(b)}{(e^2 - b^2)^{1/2}} \, db
\]  
(14)
and the energy release rate or crack extension force follows as
\[
G = (1 - \nu^2) K^2 / E
\]  
(15)
where \( b \) is the distance along the crack, \( \sigma(b) \) is the normal stress distribution, as deduced from Huber’s solution given in Appendix 1, and \( K \) and \( G \) above are defined per unit width of crack front.

Equations (14) and (15) are standard results from fracture mechanics. The principal approximation made here is to simplify the geometry of the crack by plane approximation (i.e. neglect crack curvature). Once \( G \) has been determined from equation (15), the condition for crack extension for brittle solids can be taken to be the Griffith condition
\[
G = 2 \Gamma
\]  
(16)
where \( \Gamma \) is the energy of unit area of the new fracture surface. Alternatively, if a chemical environment reacts with stress bounds at the tip of the growing crack, then the fracture criterion may be expressed in terms of rate dependent equations, leading to a kinetic equation for the rate of crack growth of the form \( \dot{e} = \psi(G \text{ or } K) \), where the fracture parameters determine the crack velocity, i.e. the crack speed is some function of either the energy release rate \( G \) or the stress intensity factor \( K \). In each of these formulations, it has been assumed that crack growth is sufficiently slow that the influence of material inertia can be neglected. For crack speeds approaching the speed of Rayleigh waves in an elastic medium,
relation (15) should have a multiplicative factor on the right hand side, which depends upon the crack speed and the elastic wave speeds of the body (see e.g. Atkinson and Eshelby\textsuperscript{14} and Freund\textsuperscript{15}). For slow crack speeds, this factor does not differ much from unity.

In the computer simulation of Lawn \textit{et al.},\textsuperscript{7} they found that they had to use a ‘pseudo’ value of Poisson’s ratio of $v=0.33$ to get good agreement with the observed angle of the cone crack $\psi=22^\circ$ for soda lime glass even though $v=0.20$–$0.25$ is more representative of glass (the correct value of $v$ gave a computed crack angle of $>30^\circ$). It is possible that a different criterion for the crack trajectory, e.g. $K_{II}=0$ (no mode II stress intensity factor), might have given a more accurate result for the crack trajectory, but this does not seem to have been investigated. The conclusion of the above-mentioned authors seems to be that the computer model is good for predicting trends, and their choice of a ‘pseudo’ Poisson’s ratio could be used to calibrate the model. However, more recently, Kocer and Collins\textsuperscript{16} modelled the problem using finite elements and growing the crack incrementally in the direction that maximises the energy release rate at each calculation step. They found that the resulting predictions of crack angle were in excellent agreement with the experimental observations using the real Poisson’s ratio of the material, and then concluding that maximising the release of strain energy is an appropriate criterion to predict crack path.

\textbf{Median vents}

For the median crack system, the situation shown in Fig. 6 for sharp indentors has been considered by Lawn and Fuller.\textsuperscript{8} They treat the situation of the well developed median crack that has propagated to the surface taking up the approximate half penny configuration of Fig. 6. This well developed crack is assumed to have been driven by the component of the indentation force normal to the median plane. Designating this force as $P_1$ for smooth contact, it can be resolved in terms of the half angle $\psi$ of the indenter to give

$$P_1 = \frac{P}{2 \tan \psi} \tag{17}$$

For rough contact, they modify this equation to

$$P_1 = \frac{P}{2 \tan \psi_1} \tag{18}$$

with $\psi_1 = \psi \pm \arctan \mu$ \tag{19}

where $\mu$ is the coefficient of sliding friction, the plus sign refers to the loading half cycle and the minus sign to the unloading half cycle. In this sense, contact friction can be regarded as either ‘blunting’ (loading) or ‘sharpening’ (unloading) the indenter.

The stress intensity factor for a crack loaded in such a way (neglecting the free surface) is evaluated as

$$K = \frac{2P_{1}}{(\pi c)^{3/2}} \tag{20}$$

Noting that $G=(1-v^2)K^2/E$ (for plane strain), the corresponding equation for the energy balance $G=2\Gamma$ (with $\Gamma$ the fracture surface energy) gives

$$\frac{P^2}{c^3} = \frac{2\Gamma E}{\kappa} \tag{21}$$

where the dimensionless constant $\kappa$ is

$$\kappa = \frac{1}{\pi^3} \frac{1-v^2}{\tan \psi_1} \tag{22}$$

Thus, the unknown proportionality constant in expression (6) has been determined for this case. Lawn and Fuller\textsuperscript{8} make comparisons with experiments on soda lime glass, which seem to justify the $P^2/c^3$ relationship of equation (21) for a variety of indentor angles. These data give moderate agreement with equation (19) with the angle of friction $\mu$ taken to be zero.

In an earlier paper, Lawn and Swain\textsuperscript{9} have given a more detailed fracture mechanics analysis of median vent formation beneath point indentors using the Boussinesq stress field given in Appendix 2. They note, in particular, that for $v=0.25$, the three principal stresses $\sigma_{11} \geq \sigma_{22} \geq \sigma_{33}$ nearly everywhere in the field and that both $\sigma_{11}$ and $\sigma_{33}$ are contained within planes of symmetry through the normal load axis with $\sigma_{11}$ everywhere tensile and $\sigma_{33}$ everywhere compressive. Thus, the tensile component of the point indentation field might well be large enough to sustain a brittle crack. The value of Poisson’s ratio is critical as far as the tensile stress is concerned: at $v=0.5$, the tensile component in the Boussinesq field disappears, and as $v$ varies between 0.2 and 0.5, the materials generally tend to vary from highly brittle to highly ductile. The fracture mechanics calculations of Lawn and Swain\textsuperscript{9} did not, however, bear out the similarity relation (21), and this is probably due to the fact that their calculations modelled the crack as planar, and the distribution of the tensile stress produced by Boussinesq field was thus not correctly distributed over the crack plane.

In a more recent paper, Lawn \textit{et al.}\textsuperscript{17} have reconsidered this problem by arguing that the complex elastic–plastic stress field beneath the indenter can be resolved into its elastic and residual components. The elastic component enhances downward median extension during the loading half cycle, but since it is reversible,
these median vents tend to close during unloading. The residual stress component is assumed to provide the primary driving force for crack configuration in the final stages of evolution, where the crack tends to the near half penny configuration, which was modelled in equation (21). They adopt the hypothesis that the origin of the irreversible field lies in the accommodation of an expanding plastic hardness impression by the surrounding elastic matrix. The resulting plastic zone is then considered as the source of an effective outward residual force on the crack. Assuming a penny-like crack geometry and various approximations to the plasticity analysis, they arrive at the following expression for the stress intensity factor $K_c$ produced by this residual force

$$K_c = \frac{Z_c P}{c^{3/2}}$$

(23)

where

$$Z_c = \frac{\xi_c (\phi)}{H} \left( \frac{E}{\rho} \right)^{1-m} \left( \cot \phi \right)^{3/3}$$

(24)

with $m=1/2$ ($m=2/5$ is perhaps a better approximation), and $\xi_c (\phi)$ is a dimensionless term independent of indentor/specimen system introduced to allow for the effects of the free surface in their approximate fracture mechanics analysis. The angle $\phi$ is the angle between the radius drawn from the point of contact of the indentor to the position on the crack front and the vertical; $\xi_c (\phi)$ is a slowly varying function of value near unity with its minimum at $\phi=0$ (median vent orientation, i.e. vertical) and maximum at $\phi=\pm 90^\circ$ (radial orientation). Essentially, this factor occurs in deriving the approximate stress intensity factor of a half penny crack with a free surface and the residual force loading at the centre.

It is worth recalling that for indentors that leave geometrically similar impressions in homogeneous specimens at all loads, the mean indentation pressure remains invariant and provides a measure of the hardness of the material, i.e.

$$H = \frac{P}{\pi a^2}$$

(25)

where $a$ is the radius of the surface hardness impression, $P$ is the indentation load and $a=1$ for axial symmetry.

Comparing expression (23) with expression (20), we see that although the two expressions have the same dependence on crack length $c$ and applied load $P$, the other factors are different. The dependence on the half angle of the indentor $\psi$ is similar, but relation (24) also involves the ratio of hardness to modulus, an expression that does not occur in the simpler analysis leading to equation (20).

The contribution to the crack driving force due to the elastic matrix itself on the loading cycle can be evaluated in terms of the prior elastic contact stresses over the crack plane at $R=b$ ($b$ is the radius of the plastic zone) (cf. Fig. 6). For this elastic field, the Boussinesq point load field of Appendix 2 is a reasonable approximation for $R>a$. As shown in Appendix 2, it has the form

$$\sigma (R,\theta) \approx g(\theta) \frac{P}{R^2}$$

(26)

where $g(\theta)$ appropriate to stresses normal to the median plane is strongly varying, changing from positive (tensile) at $\theta=0$ to negative (compressive) at $\theta=\pm 90^\circ$.

The stress intensity factor for a half penny crack subjected to radially distributed stresses over $b<R<c$ is approximated by

$$K = f(\phi) \left( \frac{2}{\rho} \right)^{1/2} \int_b^c \frac{r \sigma (r)}{(c^2-r^2)^{1/2}} dr$$

(27)

Formula (27) thus approximates somewhat the effect of $g(\theta)$ variation in equation (26). Ignoring this and substituting for equation (26) in equation (27) gives an elastic stress intensity factor of the form

$$K_e = \frac{P}{c^{3/2}}$$

(28)

where in the limit $c\gg b$

$$Z_e = \xi_c (\phi) \ln (2c/b)$$

(29)

Because of the $c/b$ dependence of equation (29), expression (28) deviates from the similarity that we deduced in equations (6b), (21) and (23). However, because of the $\ln (c/b)$ dependence of equation (29), this variation is not so great. Furthermore, Lown et al. argue that $K_e$ is secondary in importance to $K_c$ in determining the ultimate crack configuration.

The condition for equilibrium growth of these cracks is obtained by equating the net stress intensity factor $K$ to the toughness $K_C$ (this can be related to $\Gamma$ used previously for example in plane strain; since $G=(1-v^2)K^2/E$, one would have $K_C=2\pi E/(1-v^2)$). Thus, during the loading/unloading event, one would have $K=K_C+K_e=K_C$, and care must be taken to realize that equation (28), being the elastic loading component, is reversible, whereas the residual term of equation (23) is irreversible. Since we are interested in what happens after complete unloading of the crack having then attained its full half penny configuration, it seems adequate to use equation (23) and derive

$$c = \frac{P_{\text{max}}}{K_C} \left( \frac{c}{b} \right)^{3/13}$$

(30)

as the complete crack dimension $P_{\text{max}}$ being the maximum load.

For soda lime glass, the evolution of the median/radial cracks has been studied experimentally by the abovementioned authors.

**Lateral crack system**

Marshall et al. have described the mechanics of lateral crack propagation in a sharp indentor contact field by assuming that the driving force for fracture has its origin in the residual component of the elastic–plastic field, which becomes dominant as the indentor is unloaded. The picture is as shown in Fig. 4. A residual tensile stress develops at the nucleation centre near the base of the deformation zone so that a residual driving force $P_t$ acts on the crack as the indentor is withdrawn (Fig. 4). In the absence of reversed plasticity within the central zone, $P_t$ reaches its maximum at full contact loading and persists at this value on removal of the indentor. The net mouth opening force for the system is expressed as the difference between an irreversible residual (opening) component and a reversible elastic (closing) component.

The above authors claim that experimental observations indicate that lateral cracks initiate as the diminishing applied load approaches zero. The crack geometry is
taken to be penny-like but with the complication of an adjacent, parallel free surface.

We summarise here the results of the approximate analysis given in the paper of Marshall et al.\textsuperscript{18} The thickness of the material above the crack plane is identified with the depth of the plastic zone. For the residual impression formed at peak indentation load $P$

$$a = \left( \frac{P}{2E\tilde{H}} \right)^{1/2}$$  \hspace{1cm} (31)

where $\tilde{H}$ is defined as the hardness, which is an invariant for geometrically similar indentations, and $\zeta_0$ is a geometrical constant ($\zeta_0 = 1$ for axisymmetry as noted earlier). Calculation of the elastic-plastic boundary based on the model of an internally pressurised spherical cavity gives

$$\frac{b}{a} = \left( \frac{E}{\tilde{H}} \right)^{1/2} (\cot \psi)^{1/3}$$  \hspace{1cm} (32)

[A more accurate model could replace $(E/\tilde{H})^{1/2}$ by $(E/\tilde{H})^{25/14}$.]

Using equations (31) and (32) together gives for the thickness of the material above the crack plane

$$h \approx b \approx \left( \frac{EP}{H} \right)^{1/2} (\cot \psi)^{1/3}$$  \hspace{1cm} (33)

An expression for the equilibrium crack size as a function of applied load for large contact loads is given as

$$c = \left[ \frac{\zeta_4}{A^{1/2}} (\cot \psi)^{7/6} \left( \frac{E}{H} \right)^{1/4} \frac{K_H}{C_1 H^{1/4}} \right]^{1/2} P^{5/8}$$  \hspace{1cm} (34)

where $\zeta_4$ is a dimensionless constant independent of the material/indentor system. The factor $A$ is a constant depending on how the lateral crack system is affected by the other crack system, i.e. the radial crack system discussed in the median/radial section. If the lateral cracks are much larger than the radial cracks, $A$ is taken as $A = 3(1 - \nu^2)/(4\nu)$; on the contrary, if the lateral cracks are smaller than the radial cracks, a quarter plate approximation is used with $A = 3/4$. Note that for chipping to take place, the lateral vents would deviate to intersect the free surface, as shown in Fig. 4. However, the approximations leading to equation (34) assume that the crack is propagating horizontally parallel to the free surface. Furthermore, it is assumed that the cracks are stable in equilibrium, hence the appearance of the critical stress intensity factor (toughness) $K_c$ in equation (34). Comparing equation (34) with equation (8b) and noting that $P = P$, the load peak, we see that the dependence on $H$ is similar but the dependence on $P$ has changed from $P^{3/2}$ to $P^{5/8}$.

Note that the crater volume produced by the lateral cracks can be estimated as (cf. Fig. 4)

$$V = \pi \tilde{c}^2 h$$  \hspace{1cm} (35)

and hence from equations (33) and (34)

$$V = \pi \frac{\zeta_4}{A^{1/2}} (\cot \psi)^{7/6} \frac{E^{5/4}}{K_c H^{1/4}} P^{7/4}$$  \hspace{1cm} (36)

### Applications

In the section on ‘Indentation fracture’, we have described the main features of contact fracture in a ‘broad brush approach’. Here, we describe some recent (and not so recent) applications; these are merely a sample of a very large literature on the subject. We begin with an example of sensing subsurface flaws (or fracture created by contact) by the surface displacement method. Here, a heat source is produced by a laser, which heats up the subsurface and any defects there. A second laser running along at a prescribed distance behind measures the surface displacement and hence images the subsurface. This method was first introduced by Ameri et al.\textsuperscript{19}, Martin and Ash\textsuperscript{20} and Martin et al.\textsuperscript{21} and Atkinson and Martínez-Esnaola\textsuperscript{22} provided a suite of calculations whereby this thermal field interacted with surface and subsurface cracks, producing surface displacement due to the thermal expansion of the material/crack interaction. In particular, these calculations indicated the optimal spacing of the two lasers. This is just one example of how subsurface imaging can be produced. A technique for controlled crack growth and brittle fracture toughness determination has been suggested by Martin-Meizoso et al.\textsuperscript{23} in which they use a dual tip indenter to extend cracks previously generated by a Vickers indentation on soda lime glass. They show that controlled crack growth is feasible.

To describe the precise prediction of plastic flow patterns during indentation from the point of view of microscopic flow and complex dislocation interactions, Brown\textsuperscript{24} has presented a model for a wedge shaped indenter, and Czeczki and Brown\textsuperscript{25} have adapted this for a cylindrical indenter. They use a slip circle construction and assume a rigid plastic flow field and also apply this construction to a spherical indenter. However, the present work is confined to describing hardness rather than contact fracture, although the indentation flow behaviour will ultimately influence the contact fracture behaviour.

We discuss below a situation that affects the integrity of silicon wafer manufacture and show how some of the issues described in the section on ‘Indentation fracture’, such as both lateral and median cracks, can occur and describe a number of non-destructive techniques for determining them.

### Practical example

The original theory of Hertzian fracture in isotropic elastic materials was extended by Lawn\textsuperscript{26} to the case of brittle single crystals, analysing the inhomogeneous stress field and crack paths developed in the indentation of a flat surface with a spherical indenter. He found that the Hertzian crack passes through four equilibrium stages before reaching its fully developed length, which is experimentally confirmed in indentation tests on single crystals of silicon.

A compilation of indentation results on silicon crystals for several indenter types and a large range in load values covering elastic, plastic and fracture behaviours has been reported by Armstrong et al.\textsuperscript{27} In the case of sharp indenters, they attribute crack initiation to dislocation reaction mechanisms that develop during microslip deformation interaction, in agreement with the TEM results reported for damage of silicon by various contact testing experiments.

The role of dislocations in indentation (and nanoindentation) processes has been analysed by Chaudhri,\textsuperscript{28} mainly based on elastic behaviour, as well as the phase

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure.jpg}
\caption{Schematic of the indentation process.}
\end{figure}
transformation of silicon associated with stresses generated during indentation hardness experiments. Here, we describe briefly a practical example aimed at understanding silicon wafer handling damage.

In order to replicate defect generation in real processes, controlled damage was introduced on initially defect free silicon wafer samples using nanoindentation with a Berkovich tip. The tested samples consisted of square pieces of 20 × 20 mm size. They were prepared by first cleaving a 300 mm silicon wafer with (100) surface crystallographic orientation along the [011] direction indicated by the notch (see Fig. 7) and then creating the little squares in the figure (also using cleavage planes). Loads between 100 and 200 mN were applied at rates of 10 mN s⁻¹ using a Nanoindenter II (Agilent, formerly Nano Instruments Inc.) with a Berkovich diamond tip. The indents were oriented so that one of the imprint sides was parallel to the [011] direction.

Direct observation of the crack systems beneath the indents was performed using a Quanta three-dimensional (3D) dual beam (FEI) focused ion beam (FIB) system. This gives an idea of the type and extension of

7 Scheme of indented wafers and square samples prepared (highlighted in green): location of indents in wafers is also shown with one side of Berkovich tip parallel to [011] direction (note that scale of imprint is not real)

8 Sequence of field emission gun SEM images from top–down fibbing on 110 mN Berkovich indent on Si, showing coexistence of radial and median cracks close to surface and further development of median cracks under indentor imprint
damage generated by nanoindentation. The micrographs obtained from top–down fibbing, and shown in Fig. 8, suggest a double crack system, i.e. radial and median cracks, coexisting under indents, 100 mN (see Gorostegui-Colinas et al. 29). Radial (or Palmqvist) cracks emerge from the corners of the indentor imprint and have a maximum depth on the order of 0.5 μm. These close surface cracks have been observed in brittle materials and sharp indentors at low loads (see, for instance, Tang et al. 30). In addition, median cracks originate beneath the remaining plastic region under the indentor imprint at higher loads and propagate along the median axis upon unloading.

Furthermore, direct observation through cross-sectional fibbing indicates that, in addition to the mentioned crack systems, lateral cracking appears under the indentor imprint and have a maximum depth on the order of 0.5 μm. These close surface cracks have been observed in brittle materials and sharp indentors at low loads (see, for instance, Tang et al. 30). In addition, median cracks originate beneath the remaining plastic region under the indentor imprint at higher loads and propagate along the median axis upon unloading.

Figure 11 shows a schematic of the finite element model designed for stress computations and fracture analysis of the indentations. The simplest model consists of a 48 × 48 × 96 μm piece of Si, which is assumed to behave as an anisotropic elastic material but plastically isotropic. The central part is discretised using a structured mesh of eight-node hexahedral elements of size 0.25 μm. A finer meshing of the silicon piece using submodelling techniques was also used in combination with cohesive zone models to investigate the initiation and propagation of cracks during indentation. In the latter model, cohesive elements of zero thickness were placed in the prospective planes of cracking development. A triangular traction–separation law was used, characterised by the peak stress $\sigma_{\text{max}}$ and the separation energy (area below the traction–separation law) $\Gamma_0$ as the main parameters (the initial slope of the traction–separation load will not be discussed here). Based on literature data (see Sopori 31), the values $\sigma_{\text{max}} = 1200$ MPa and $\Gamma_0 = 4$ J m$^{-2}$ were used in the calculations although better approximations could be obtained if one takes into account the anisotropic toughness properties of the silicon. Details of the numerical analyses performed can be found in the study of Gorostegui-Colinas et al. 29

One way of validating the models is through micro-Raman spectroscopy experiments carried out in silicon wafers. Details of this experimental technique can be found in Allen et al. 32 The Raman measurements agree quite well with the frequency shifts resulting from the stress field predicted in the finite element model simulations, as shown in Fig. 12. The discrepancies can be attributed to the lack of consideration of the phase transformations in silicon during unloading. Note also that the resolution of the Raman measurements is limited in the central part of the indent imprint due to the inclination of the surface.

Figure 13 shows contour plots of the damage parameter associated with the cohesive zone approach and a 3D reconstruction of a vertical real crack. The results indicate that during unloading of the indentation process fracture continues propagating. The shape of the crack agrees qualitatively with the experimental results.

**Conclusions**

We have concentrated in the section on 'Indentation fracture' on outlining the consequences of the inhomogeneous stress field produced by the indentor, the main theme being that tensile stresses can initiate cracks and subsequently sustain crack propagation. We have emphasised the crack propagation stage since such macrocracks lead to chip formation. However, if
we consider a material like rock, which is such an inhomogeneous material, there will inevitably also be a number of microcracking events. The existence of microscopic flaws is, of course, a well known ingredient in theoretical models that attempt to explain the failure of rock under compression. Costin and Holcomb\textsuperscript{33,34} and Costin\textsuperscript{35} have considered microcrack models for the deformation and failure of brittle rock. They represent the rock by an essentially elastic constitutive equation with material behaviour due to microcrack growth accounted for by the inclusion of an internal state variable, which is a measure of the crack state. The form of the evolutionary equation for this crack state parameter is determined from fracture mechanics analysis of single cracks and experimental results. This model has been used by the above authors to simulate uniaxial and triaxial compression tests and comparisons made with laboratory tests on westerly granite. The idea that a microcracked solid can be represented by an equivalent continuum model is, of course, a natural consequence of modern continuum mechanics, but imposed on any such model is the deliberate inhomogeneous stress

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Comparison of 2D Raman spectrum map obtained experimentally (left) and prediction based on stress field resulting from finite element model simulations (right)}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{a at maximum indentation load; b after indentation process (broken material has been removed from plot for clarity); c 3D reconstruction of vertical experimental crack in same plane}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Crack in vertical plane: contour plots of generated damage}
\end{figure}
field imposed by the contacting indenter, and it is this action that most concerns us here. If we did have a good continuum model of the underlying material microstructure, then we would calculate the resulting inhomogeneous stress field produced by the indenter and then rationalise the resulting crack field as outlined in the section on 'Indentation fracture'. In the section on 'Applications', this process has been carried out effectively by means of a variety of techniques: finite element stress analysis, crack growth analysis plus Raman spectroscopy, TEM, etc. to rationalise a practical problem.

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Appendix 1

Hertzian stress field

The Hertzian stress field has the property of geometrical similarity when all stress components are normalised to the mean contact pressure \( p_0 = P/(\pi a^2) \) and all spatial coordinates normalised to the contact radius \( a \). The stresses in the OXZ plane (see Fig. 5) are given by Huber\(^2\)

\[
\frac{\sigma_{xx}}{p_0} = \frac{1}{2} \left( \frac{a}{r} \right)^2 \left[ 1 - \frac{(z/a)^2}{(1 + \epsilon)^2} \right] + \frac{5}{2} \frac{z}{(1 + \epsilon)^2} \left[ 1 + \frac{z}{(1 + \epsilon)^2} \right]
\]

\[
\frac{\sigma_{zz}}{p_0} = \frac{1}{2} \left( \frac{a}{r} \right)^2 \left[ 1 - \frac{(z/a)^2}{(1 + \epsilon)^2} \right] + \frac{5}{2} \frac{z}{(1 + \epsilon)^2} \left[ 1 + \frac{z}{(1 + \epsilon)^2} \right]
\]

where \( r^2 = x^2 + z^2 \) and

\[
u = \frac{1}{2} \left[ (x^2 + z^2 - a^2) + \left( (x^2 + z^2 - a^2)^2 + 4a^2z^2 \right)^{1/2} \right]
\]

The principal normal stress across the crack path (at an angle \( \alpha \) to the horizontal) is

\[
\sigma_N(x, z) = \sigma_{xx} \sin^2 \alpha + \sigma_{zz} \cos^2 \alpha - 2\sigma_{xz} \sin \alpha \cos \alpha
\]

The angle \( \alpha \) between the crack path and the specimen surface is found from

\[
\tan 2\alpha = -2 \frac{\sigma_{xz}}{\sigma_{xx} - \sigma_{zz}}
\]

Appendix 2

Boussinesq problem

The stress field in an isotropic elastic half space subjected to a normal point load \( P \) (the Boussinesq problem) can be written in terms of the curvilinear coordinates of Fig. 14 as

\[
\sigma_{xx} = \frac{P}{\pi R^2} \left[ 1 - \frac{2v}{4 \sec^2 (\phi/2) - \frac{3}{2} \cos \phi \sin^2 \phi} \right]
\]

\[
\sigma_{yy} = \frac{P}{\pi R^2} \left[ 1 - \frac{2v}{2 \left[ \cos \phi - \frac{1}{2} \sec^2 (\phi/2) \right]} \right]
\]

\[
\sigma_{zz} = \frac{P}{\pi R^2} \left( -\frac{3}{2} \cos^3 \phi \right)
\]
Two of the principal normal stresses, $s_{11}$ and $s_{33}$, are contained in the symmetry plane $h=\text{constant}$, with their angles with the specimen surface given by

$$\tan 2\alpha = \frac{s_{rr}}{s_{zz} - s_{rr}}$$

The third principal normal stress $s_{22}$ is everywhere perpendicular to the symmetry plane. The principal directions are labelled such that $s_{11} > s_{22} > s_{33}$ generally ($\sigma > 0$ for tension). Then, noting that $s_{zz} \leq 0$ for all $0 \leq \phi \leq \pi/2$, we have

$$s_{11} = s_{zz} \cos^2 \alpha + s_{zz} \sin^2 \alpha - 2s_{rz} \sin \alpha \cos \alpha$$

$$s_{22} = s_{\phi \phi}$$

The principal shear stresses are given by

$$s_{13} = \frac{s_{11} - s_{33}}{2}$$

$$s_{12} = \frac{s_{11} - s_{22}}{2}$$

$$s_{23} = \frac{s_{22} - s_{33}}{2}$$

inclined at $\pi/4$ to the principal directions. In addition, the magnitude of the component of hydrostatic compression is

$$p = -\frac{s_{11} + s_{22} + s_{33}}{3}$$

Figure 15 shows a comparison of the $s_{00}$ stress term for Boussinesq and Hertzian fields and their divergence in the vicinity of the contact zone (see for example Lawn and Wilshaw36): at $z=0$, $s_{00} \to \infty$ in the Boussinesq field, whereas $s_{00} \to -1.25P_0$ in the Hertzian field.