1 Introduction

Successful operation of modern machinery over long periods of time is critically dependent on the use of appropriate lubricants at contacting interfaces. As discussed before, intervening layers as small as a few molecular diameters are already capable of producing major improvements in tribological system performance. Major additional improvements in tribological performance are possible through judicious application of lubrication engineering principles. The many available lubricant choices are associated with a few specific lubrication regimes. These are in turn differentiated by the film associated film thickness as follows:

- **Hydrodynamic Fluid Film.** The layer completely separates and prevents direct contact of the solid surfaces involved. The film thickness is several times larger than the magnitude of the composite standard deviation of surface heights of the contacting surfaces, usually of the order of 100 micrometers.

- **Elastohydrodynamic.** The layer is severely compressed and thinned by the applied load. Direct contact is still prevented but contacting solids deform elastically because of the high film pressure. The film thickness is only slightly larger than the magnitude of the composite standard deviation of surface heights of the contacting surfaces, usually of the order of 1 micrometer.

- **Transition or Mixed Lubrication.** The film thickness becomes of the order of the surface asperities (i.e. of the order of 0.1 micrometer) and intermittent contact is obtained.

- **Boundary Lubrication.** The film can be as small as a single adsorbed layer. Intermittent solid contact may take place.

The friction behavior under the above regimes is clearly shown on a *Stribeck curve* which is a plot of friction coefficient values $\mu$ in a logarithmic scale versus the parameter $\eta V/P$ where $\eta$ is the *dynamic viscosity of the film material*, $V$ is the relative velocity and $P$ is the load per unit projected area. The curve shows a minimum value for $\mu$ which can be as low as 0.001 corresponding to the thinnest hydrodynamic films. Values of $\mu$ increase rapidly for thinner films as first intermittent contact and then more intensive direct contact of the
surfaces is obtained. The friction coefficient also increases as the lubricating film gets thicker due to the effect of viscosity.

2 Fluid Film Lubrication

This section introduces basic concepts of lubricant action and then focuses on the study of the hydrodynamic fluid film lubrication regime.

2.1 Lubricant Types and Characteristics

According to their physical characteristics lubricants are classified as follows:

- **Mineral Oils.** A complex mix byproduct of fractional distillation of crude oil.
- **Synthetic Oils.** Produced by polymerization of low molecular weight hydrocarbons.
- **Greases.** Mixtures of lubricating oils and thickeners obtained by adding alkali and fatty acid to oil.
- **Boundary Lubricants.** Molecules with strong affinity towards the surface being lubricated.
- **Solid Lubricants.** Layered and non-layered lattice solids; fullerenes.

In fluid film hydrodynamic lubrication both mineral and synthetic oils are commonly used. Key properties of these lubricants which must be considered in engineering design include

- **Viscosity:** dynamic and kinematic.
- **Physical properties:** density, conductivity, specific heat, surface tension, refractive index, additive compatibility and solubility, impurity content.
- **Stability:** pour, cloud, flash and fire points; volatility, oxidation rate.

Viscosity is a most important property of fluid lubricants. It can simply be regarded as a measure of the internal molecular friction in the fluid resisting shearing. Consider a fluid contained between two large parallel flat plates perpendicular to the $y$-axis. As the upper plate moves at constant velocity $U$ along the $x$-direction, a drag force is exerted on the lower plate due to the viscosity of the fluid. If the drag force per unit area is the *shear stress* $\tau$ and the *velocity gradient* in the fluid in the direction of the normal to the plates is $du/dy$, the *dynamic viscosity* $\eta$ is defined as

$$\eta = \frac{\tau}{du/dx}$$
The kinematic viscosity $\nu$ is obtained from $\eta$ after dividing by the fluid density $\rho$, i.e. $\nu = \eta/\rho$. The dynamic viscosity is measured in Pascal-second (Pa-s) or in centiPoise (cP) where 100 cP = 0.1 Pa-s. The kinematic viscosity is measured in $m^2/s$ or in centiStokes (cS) where 100 cS = 0.0001 $m^2/s$.

The dynamic viscosities of lubricating oils are in the range between 0.01 and 1 Pa-s. Numerical values of kinematic viscosities are between one hundred and one thousand times smaller.

Viscometry is the set of experimental techniques used for the measurement of fluid viscosity. The most commonly used configurations are:

- Capillary viscometer. Based on the measurement of the time required for a given amount of fluid to flow through a capillary.
- Rotating cylinder viscometer. Based on the measurement of the force required to shear the fluid contained in the gap between two concentric cylinders.
- Cone on plate viscometer. Based on the measurement of the shear torque on a cone making point contact with a plane and containing the lubricating fluid in the gap.

The viscosity is sensitive to temperature and pressure changes. Oil viscosity drops rapidly as temperature increases while it increases with increasing pressure. Various empirical expressions have been employed to describe the temperature dependence of viscosity. A most accurate formula, widely used in engineering calculations is:

$$\eta = \eta_0 \exp(\beta[\frac{1}{T} - \frac{1}{T_0}])$$

A commonly used formula to express the variation of viscosity with pressure is

$$\eta = \eta_0 \exp(\alpha p)$$

### 2.2 Fluid Dynamics of Lubricating Films

The flow of lubricants obeys the basic laws of fluid mechanics, namely, the equation of conservation of mass (equation of continuity) and the momentum conservation equations (Navier-Stokes). The assumption of incompressibility is perfectly adequate in most cases.

The equation of continuity for an incompressible fluid in Cartesian coordinates is

$$\text{div} \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

The equation of Navier-Stokes for a Newtonian fluid also in rectangular Cartesian coordinates ($x_1, x_2, x_3$) is the statement of the balance of momentum along each of the three the $x_i$ directions

$$\rho \frac{\partial v_{x_i}}{\partial t} + \mathbf{v} \cdot \nabla v_{x_i} = -\frac{\partial p}{\partial x_i} + \eta \nabla^2 v_{x_i} + \rho g x_i$$
for \( i = 1, 2, 3 \). Here \( t \) is time, \( \mathbf{v} = (v_{x_1}, v_{x_2}, v_{x_3}) \) is the velocity field vector and \( \mathbf{g} = (g_{x_1}, g_{x_2}, g_{x_3}) \) is the gravitational acceleration vector.

For the analysis of fluid flow in lubricating films the following assumptions are commonly made.

- Negligible body forces.
- Steady state conditions.
- Negligible inertia forces.
- Constant pressure through film.
- Laminar flow.
- Newtonian fluid.
- Constant fluid density.
- No slip at boundaries.
- Rigid and smooth solid surfaces.
- Constant viscosity through film.

The equations resulting from the introduction of the above assumptions into the original governing equations of fluid mechanics constitute the statement of lubrication theory.

The governing equations of fluid mechanics (i.e. continuity and motion) can be combined under the assumptions of lubrication theory to yield a single equation to compute the pressure inside the film (Reynolds equation). Consider a lubricating film constrained between two solid surfaces. In a Cartesian coordinate system let the \( z \)-axis be located along the direction of the film thickness \( h(x, y) \), while the span of the liquid layer on the \( x - y \) plane is much larger than its thickness. Moreover, let the fluid motion be driven by the relative velocity \((U, V)\) and be restricted to the \( x - y \) plane.

With the introduced assumptions, the equations of motion become

\[
\frac{\partial p}{\partial x} = \frac{\partial}{\partial z} \left( \eta \frac{\partial u}{\partial z} \right)
\]

\[
\frac{\partial p}{\partial y} = \frac{\partial}{\partial z} \left( \eta \frac{\partial v}{\partial z} \right)
\]

where \( u = u(x, y, z) \) and \( v = v(x, y, z) \) are the \( x \) and \( y \) components of velocity, respectively.

Since at any given \((x, y)\) location, the pressure and the pressure gradient have fixed values, independent of \( z \), the above equations can be readily integrated subject to appropriate boundary conditions to yield the values of the velocity components at that location.
For boundary conditions at $z = 0$ and $z = h(x, y)$, consider the following

\[
\begin{align*}
    u(x, u, 0) &= U_0 \\
    v(x, y, 0) &= V_0
\end{align*}
\]

and

\[
\begin{align*}
    u &= u(x, y, h) = U_h \\
    v &= v(x, y, h) = V_h
\end{align*}
\]

Solving for $u$ and $v$ yields

\[
\begin{align*}
    u(x, y) &= -z \frac{(h - z)}{\eta} \frac{\partial p}{\partial x} + U_h \frac{h - z}{h} + U_0 \frac{z}{h} \\
    v(x, y) &= -z \frac{(h - z)}{\eta} \frac{\partial p}{\partial y} + V_h \frac{h - z}{h} + V_0 \frac{z}{h}
\end{align*}
\]

The (local) volume flow rates (per unit width) along the $x$ and $y$ directions, $q_x, q_y$ are then obtained as

\[
\begin{align*}
    q_x &= \int_0^h u \, dz = -\frac{h^3}{12\eta} \frac{\partial p}{\partial x} + \frac{U_0 + U_h}{2} h \\
    q_y &= \int_0^h v \, dz = -\frac{h^3}{12\eta} \frac{\partial p}{\partial y} + \frac{V_0 + V_h}{2} h
\end{align*}
\]

Now, integration the steady-state continuity equation for a constant density fluid across the gap thickness yields

\[
\begin{align*}
    \int_0^h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz &= \\
    = -U_0 \frac{\partial h}{\partial x} - V_0 \frac{\partial h}{\partial y} + \frac{\partial}{\partial x} \left( \int_0^h u \, dz \right) + \frac{\partial}{\partial y} \left( \int_0^h v \, dz \right) + (W_0 - W_h) = \\
    = -U_0 \frac{\partial h}{\partial x} - V_0 \frac{\partial h}{\partial y} + \frac{\partial}{\partial x} q_x + \frac{\partial}{\partial y} q_y + (W_0 - W_h) = 0
\end{align*}
\]

where $W_0$ and $W_h$ are the values of the $z$–component of velocity at $z = 0$ and $z = h$, respectively.

Finally, introducing the above derived expressions for $q_x$ and $q_y$ into the integrated continuity equation yields

\[
\begin{align*}
    \frac{\partial}{\partial x} \left( -\frac{h^3}{12\eta} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( -\frac{h^3}{12\eta} \frac{\partial p}{\partial y} \right) + \frac{\partial}{\partial x} \left( \frac{h(U_0 + U_h)}{2} \right) + \frac{\partial}{\partial y} \left( \frac{h(V_0 + V_h)}{2} \right) + \\
    + (W_0 - W_h) - U_0 \frac{\partial h}{\partial x} - V_0 \frac{\partial h}{\partial y} = 0
\end{align*}
\]
this is the general Reynolds equation giving the pressure $p(x,y)$ inside the lubricating film.

Considering only tangential motion such that $W_0 = U_0 \partial h/\partial x + V_0 \partial h/\partial y$ and that only the surface at $z = 0$ moves (i.e. $U_0 = U, V_0 = V, U_h = V_h = 0$), the general Reynolds equation reduces to

$$\frac{\partial}{\partial x} (h^3 \frac{\partial p}{\eta \partial x}) + \frac{\partial}{\partial y} (h^3 \frac{\partial p}{\eta \partial y}) = 6U \frac{\partial h}{\partial x} + 6V \frac{\partial h}{\partial y}$$

This is frequently known as the Reynolds equation.

Solutions to the Reynolds equation must almost always be obtained using numerical methods but exact solutions are possible for a selected number of important limiting cases.

An important simplification is obtained if side ($y$) leakage is neglected. In this case, the Reynolds equation reduces to

$$\frac{\partial}{\partial x} (h^3 \frac{\partial p}{\eta \partial x}) = 6U \frac{\partial h}{\partial x}$$

which can be integrated subject to the condition that at $x = x^*$, $dp/dx = 0$ to yield the frequently used integrated form of the Reynolds equation

$$\frac{dp}{dx} = 6U \frac{h - h^*}{h^3}$$

where $h^* = h(x^*)$.

### 2.3 Special Limiting Cases of Lubricating Flows

Various important limiting cases of the three dimensional Reynolds equation are readily obtained by introducing additional simplifying assumptions.

Consider the case of flow between parallel plates. This configuration is encountered in hydrostatic bearings. Let the plates be wide and long so that the flow is mainly along the $x$-direction and driven by the pressure gradient $dp/dx$. Here the film thickness $h$ is constant and is aligned with the $z$-axis. The Navier Stokes equations reduce to the following form of the $x$ momentum balance

$$\frac{dp}{dx} = \eta \frac{d^2 u}{dz^2}$$

Integrating twice with respect to $z$ yields

$$u(z) = \frac{1}{2\eta} \frac{dp}{dx} z(z - h)$$

this is the well known Hagen-Poiseuille equation. Furthermore, since volume must be conserved

$$q = \int_o^h udz = -\frac{h^3}{12\eta} \frac{dp}{dx}$$
So that for every (constant) value of the pressure gradient there corresponds a constant volumetric flow rate.

As a second example consider the hydrostatic thrust bearing with a circular step pad. Here the film thickness is \( h_i \) for \( 0 < r < r_i \) (the recess region) and \( h_o \) for \( r_i < r < r_o \) (the land region). While the pressure may be assume constant and equal to \( p_i \) in the recess region, the Reynolds equation in cylindrical polar coordinates in the land region becomes

\[
\frac{d}{dr} \left( r \frac{dp}{dr} \right) = 0
\]

Integrating twice and introducing the boundary conditions \( p(r_i) = p_i \) and \( p(r_o) = 0 \) yields the pressure distribution in the land as

\[
p(r) = p_i \frac{\ln(r_o/r)}{\ln(r_o/r_i)}
\]

The volumetric flow rate per unit circumferential distance \( q \) is then

\[
q = -\frac{h^3}{12\eta} \frac{dp}{dr} = \frac{h^3 p_i}{12\eta r \ln(r_o/r_i)}
\]

and the total volumetric flow rate \( Q = 2\pi rq \).

The normal load borne by the bearing, \( W_z \), is then

\[
W_z = \pi r_i^2 p_i + \int_{r_i}^{r_o} \pi r^2 \pi r dr = \frac{\pi p_i (r_o^2 - r_i^2)}{2 \ln(r_o/r_i)} = \frac{3\eta Q h^3}{h^3 (r_o^2 - r_i^2)}
\]

Several important limiting cases are also obtained for hydrodynamic bearings. Consider the flow in a tilted pad bearing which is very wide in the \( y \) direction so that film thickness is \( h(x) \) and the dominant component of velocity is along the \( x \) direction \( v_x(z) = u(z) \). In this configuration, the upper surface is tilted and fixed while the lower surface is horizontal and moves at constant speed \( U \) along the negative \( x \)-direction. The \( z \)-axis is perpendicular to the horizontal surface. Furthermore, let the film thickness at the pad inlet be \( h_i \) and \( h_o \) at the outlet and let \( h(x) \) be the linear function describing the gap opening as a function of distance \( x \) with slope \( dh/dx > 0 \). With the above additional assumptions the governing equation becomes

\[
\frac{dp}{dx} = \eta \frac{\partial^2 u}{\partial z^2}
\]

Integrating twice with respect to \( z \) and introducing the boundary conditions \( u = U \) at \( z = 0 \) and \( u = 0 \) at \( z = h \) yields the velocity profile \( u(z) \) inside the film for any point \( x \) along the pad

\[
u = \frac{1}{2\eta} \frac{dp}{dx} (z - h) + \left(1 - \frac{z}{h}\right) U
\]
The velocity must also satisfy the equation of mass conservation (continuity) expressing the constancy of the volumetric flow rate $q$, i.e.

$$q = \int_0^h udz = -\frac{h^3}{12\eta} \frac{dp}{dx} + \frac{Uh}{2}$$

The Reynolds equation is

$$\frac{\partial}{\partial x} \left( h^3 \frac{\partial p}{\partial x} \right) = 6U \eta \frac{dh}{dx}$$

while the pressure gradient is given by

$$\frac{dp}{dx} = \frac{12\eta}{h^3} \left( \frac{Uh}{2} - q \right) = 6U \eta \frac{h - h^*}{h^3}$$

Lubricating action makes the pressure inside the pad increase over and above the inlet and outlet values such that, at some point $x^*$ inside the pad, where the film thickness is $h^*$, the pressure reaches a maximum so that $dp/dx = 0$ and $q = const = Uh^*/2$. Hence, $h^*$ is the gap opening at which both $dp/dx = 0$ and $p = p_{max}$.

For a fluid at pressures $p_0$ at both inlet and outlet of the slider the pressure gradient equation can be integrated to give the pressure inside the slider as

$$p(x) = p_0 + \frac{6\eta U}{dh/dx} \frac{(h_i - h)(h - h_o)}{h^2(h_i + h_o)}$$

Knowing the specific form of the function $h(x)$ for the film thickness, the pressure gradient equation can be readily integrated to obtain the pressure distribution $p(x)$ inside the film for a variety of important bearing configurations.

Specifically, for the linear pad bearing with

$$h(x) = h_o (1 + m (1 - \frac{x}{L}))$$

where $m = (h_i/h_o) - 1$ and $L$ is the length of the pad, use of the boundary conditions $p(0) = p(L) = 0$ yields

$$p = \frac{6\eta UL}{h_o^2} \frac{m (x/L)(1 - x/L)}{(2 + m)(1 + m - mx/L)^2}$$

And for the exponential pad extending from $-B < x < 0$ with thickness given by $h(x) = h_o \exp(-\alpha x)$, where $\alpha = (1/B) \ln(h_i/h_o)$ the result is

$$p(x) = \frac{3U \eta}{h_o^2 \alpha} \left[ \exp(2\alpha x) - \exp(3\alpha x) \right]$$

Now for the plane inclined pad bearing extending from $-B < x < 0$ with thickness given by $h(x) = h_i - \frac{h_i - h_o}{B} x$, the result is

$$p(x) = \frac{6U \eta B}{(h_i/h_o - 1) (h_i + h_o)} \frac{h_i}{h} \left[ \frac{1}{h_i} - \frac{1}{h_o} \right] \left[ \frac{1}{h_o} - \frac{1}{h_i} \right]$$
As a third example, consider the Rayleigh step bearing extending from $-B < x < 0$ with thickness given by $h(x) = h_i$ for $x$ over the span $B_i$ and $h(x) = h_o$ for $x$ over the span $B_o$ with $B = B_i + B_o$, the pressure distribution consists of two linear regions connected at the peak pressure $p^*$ given by

$$p^* = \frac{(B_i/B_o)(h_i/h_o - 1) 6U\eta B_o}{(h_i/h_o)^3 + B_i/B_o \ h_o^2}.$$

Another important case is the study of the flow in an infinitely wide journal bearing. This case is approximated when the width is at least four times as large as the radius. Journal bearings are commonly used in a wide variety of applications. The journal is a rotating shaft supported in a circular sleeve (bearing or bushing). The bearing radius $r_b$ is slightly larger than that of the journal $r_j$ with the clearance $c = r_b - r_j$. Under rotating and loaded conditions the centers of the journal and the bearing become separated by a distance $e$ called the eccentricity. As a result, the gap between the journal and the bearing varies over the circumference yielding a minimum film thickness given by

$$h_{\text{min}} = c - e = c(1 - \epsilon)$$

where $\epsilon = e/c$ is the eccentricity ratio. Furthermore, the actual gap over the circumference is well approximated by

$$h \approx c(1 + \epsilon \cos \theta)$$

In this case the integrated Reynolds equation is

$$\frac{dp}{d\theta} = 6\eta r_j^2 \omega \frac{h - h^*}{h^3} = 6\eta \omega \left( r_j^2 \frac{1}{c} \left( 1 + \epsilon \cos \theta \right)^2 - \frac{h^*}{c(1 + \epsilon \cos \theta)^3} \right)$$

where $\omega$ is the angular velocity of the journal and $d\theta = dx/r_j$.

Sommerfeld assumed periodic boundary conditions, performed a clever change of variable for the integration and obtained the following solution for the pressure distribution

$$p = p_o + 6\eta \omega \left( \frac{r_j}{c} \right)^2 \frac{6 \epsilon \sin \theta (2 + \epsilon \cos \theta)}{(2 + \epsilon^2)(1 + \epsilon \cos \theta)^2}$$

where $p_o$ is the pressure at the location of maximum film thickness. Although the pressure rise on the upstream side of the bearing is of the correct magnitude, the pressure drop on the downstream side is not in good agreement with experience.

A related important case is the study of the flow in a narrow journal bearing. The Reynolds equation for this case is

$$\frac{\partial}{\partial x} (h^3 \frac{\partial p}{\partial x}) + \frac{\partial}{\partial y} (h^3 \frac{\partial p}{\partial y}) = 6U \eta \frac{dh}{dx}$$
For a slider of width $W$ in the $y$–direction and breath $B$ along the $x$–direction, such that $W << B$, the second term on the left hand side to the above dominates giving Ocvirk'\'e equation which is readily integrated to give

$$p(y) = \frac{6\eta U}{h^3} \frac{dh}{dx} \left( y^2 - \frac{W^2}{4} \right)$$

where $W$ is the width of the slider.

As a final example, consider the case of an infinite rigid cylinder of radius $R$ moving at steady state conditions with velocity $U$ along the $x$–direction over a fixed plane surface and in the presence of a lubricating fluid. The gap between the cylinder and the flat surface is given by

$$h = h_c + \frac{x^2}{2R}$$

where $h_c$ is the minimum clearance between the surfaces located at $x = 0$. The appropriate form of Reynolds equation is

$$\frac{\partial}{\partial x} \left( h^3 \frac{\partial p}{\partial x} \right) = 6\eta \frac{dh}{dx}$$

This must be solved subject to either Sommerfeld conditions, i.e.

$$p(x \to -\infty) = p(x \to +\infty) = \frac{dp}{dx}|_{x\to-\infty} = \frac{dp}{dx}|_{x\to+\infty} = 0$$

with $p > 0$ for $x < 0$ and $p < 0$ for $x > 0$, or Reynolds boundary conditions i.e.

$$p(x \to -\infty) = p(x = x^*) = \frac{dp}{dx}|_{x\to-\infty} = \frac{dp}{dx}|_{x=x^*} = 0$$

where $x^* > 0$ is a yet to be determined location downstream from the minimum gap location.

Reynolds proposed the use of an alternative boundary condition which resulted in much better agreement with experimental results. Sometimes, the Reynolds condition is known as the cavitation boundary condition.

Solving with the later yields the maximum pressure given by

$$p_{\text{max}} = 2.15U\eta\sqrt{\frac{R}{h_c^3}}$$

Finally, the minimum clearance film thickness under load $L$ is given by

$$h_c = 4.9\frac{U\eta RB}{2L}$$

For typical values of the variables involved this yields $h_c \approx 0.01$ micrometers. Since minimum clearances observed are much larger than this and additional mechanism of film pressuring must be at work. This is the subject of elastohydrodynamic lubrication.
2.4 Petroff’s Law and Sommerfeld Number

Petroff first obtained a closed form expression for the friction coefficient in a lightly loaded journal bearing (radius $R$, breadth $b$, clearance $c$, rotational speed of shaft $N$), by equating the torque due to the load acting on the shaft to the torque required to shear the lubricant.

The torque $T$ involved in shearing the lubricant film is

$$T = (\tau A)R = \frac{4\pi^2 R^3 b\mu N}{c}$$

where

$$\tau = \eta \frac{\partial u}{\partial y} \approx \mu \frac{U}{c} = \frac{2\pi R\mu N}{c}$$

is the shear stress in the fluid and $A = 2\pi Rb$.

When a load $L$ acts on the shaft, the frictional force $F_f = \mu L$ and hence the torque is $T = F_f R = \mu LR = \mu R(2R)bP$ where $P$ is the load per unit of projected bearing area.

Equating and solving for $\mu$ yields finally

$$\mu = 2\pi^2 \eta N \frac{R}{P} \frac{R}{c}$$

The dimensionless quantity

$$S = \left(\frac{R}{c}\right)^2 \frac{\eta N}{P}$$

is the Sommerfeld number and it is useful in lubrication system design since it involves all the main design variables.

3 Elasto-Hydrodynamic Lubrication

At large loads lubricating films in converging gaps are capable of supporting much greater pressures than those estimated using standard lubrication theory. Two key effects that must be considered are

- the pressure sensitivity of the viscosity, and
- the elastic deformation of the solid surfaces.

Elastohydrodynamic (EHD) lubrication analysis accounts for the above two effects and allows determination of fluid film thickness and pressure distribution which are in good agreement with experimental measurements on heavily loaded lubricated contacts.

In heavily loaded bearings, high pressures develop inside the entrapped fluid film. Since lubricant viscosity is pressure dependent the lubricating film exhibits solid-like behavior under these conditions. The purpose of EHL analysis is to quantify these effects.
The starting point of the analysis is the incorporation of the pressure sensitivity of viscosity with the appropriate form of Reynolds equation. Then, the concept of Hertzian contact is introduced which gives the stress and strain fields as well as the deformation of unlubricated surfaces in contact under load and combined with the viscous effect.

In a pressurized liquid film, a contact area with an approximately uniform film thickness $h_0$ is created while outside the uniform thickness zone the film thickness $h$ varies approximately parabolic with distance.

Consider the analysis of the pressure profiles in lubricating films incorporating the effect of pressure on lubricant viscosity. Specifically, consider the tilted pad slider again. The pressure gradient is given by

$$\frac{dp}{dx} = 6U \eta \frac{h - h^*}{h^3}$$

The pressure dependency of viscosity can be represented by Barus equation

$$\eta = \eta_0 \exp(\alpha p)$$

where $\eta_0$ is the pre-exponential coefficient and $\alpha$ is the pressure-viscosity index. Combining the above two equations yields

$$\frac{dp}{dx} = 6U \eta_0 \exp(\alpha p) \frac{h - h^*}{h^3}$$

Introducing the reduced pressure $q$ defined by

$$q = \frac{1 - \exp(-\alpha p)}{\alpha}$$

the pressure gradient equation becomes

$$\frac{dq}{dx} = 6U \eta_0 \frac{h - h^*}{h^3}$$

Note this is identical in form to the one obtained for the pressure gradient itself when the pressure sensitivity of viscosity is neglected. Hence

$$q_{\text{max}} = 2.15U \eta_0 \sqrt{\frac{R}{h_c^3}}$$

Furthermore, the film thickness $h_0$ is calculated as

$$h_0 = 1.66(\alpha \eta_0 U)^{2/3} R^{1/3}$$

For typical values of the variables involved this yields $h_0 \approx 0.03$ micrometers. This is still small compared to observed values.

Consider now the analysis of pressure profiles in lubricating films incorporating the effect of elastic deformation of contacting solids. A common feature of full film hydrodynamic
lubrication systems is the presence of converging/diverging gaps around a minimum clearance point. Since Hertzian contacts contain converging/diverging wedges, lubrication behavior can be expected to occur under dynamic conditions.

Under the combined influence of solid elasticity and fluid viscosity pressures in the liquid film may rapidly rise up to Hertzian solid contact levels. As a result, the film thickness becomes approximately constant and equal to $h_0$ within the contact area. However, in order to maintain continuity and to compensate for the loss of viscosity towards the contact exit a constriction of the gap down to size $h_{\text{min}}$ is formed near the downstream exit. As a consequence, a pressure spike is formed and this is followed by a subsequent decay to values below the Hertz solution. The pressure on the upstream side lies also below the Hertzian value while it extends a greater distance towards the upstream direction.

For line contact to two cylinders, the gradient in reduced pressure is given by

$$
\frac{dq}{dx} = 6 \eta_0 \frac{h - h_0}{h^3}
$$

and taking $q \approx 1/\alpha$ yields

$$
q = \frac{1}{\alpha} = 6 \eta_0 \int_{h_\infty}^{h_i} \left( \frac{h - h_0}{h^3} \right) dx
$$

The above was first solved numerically by Grubin after assuming the surfaces have the deformed shape of an unlubricated contact but separated by a gap $h$ given by

$$
h = \frac{x^2}{2R} + w_z - \delta
$$

where $w_z$ is the deformation of the cylindrical surface and $\delta$ is the normal relative displacement of distant points in the two surfaces.

Neglecting the downstream film constriction yields the following approximate expression for $h_0$

$$
\frac{h_0}{R^*} = K_G \left( \frac{U \eta_0 \alpha}{R^*} \right)^{0.7273} \left( \frac{L}{BE* R^*} \right)^{-0.0909}
$$

where $L$ is the load, $E^*$ is the reduced Young’s modulus, $R^*$ is the reduced radius of curvature and $B$ the contact breadth and $K_G$ is a constant coefficient of the order of unity.

Hanrock and Dowson first solved the problem taking the exit constriction into account and obtained the following approximate formulae for the characteristic gap separations $h_0$ and $h_{\text{min}}$

$$
\frac{h_0}{R^*} = K_{HD1} \left( \frac{U \eta_0 \alpha}{E^* R^*} \right)^{0.67} \left( \frac{L}{E^* R^*} \right)^{0.53} \left( \frac{L}{E^* R^*} \right)^{-0.067}
$$

and

$$
\frac{h_{\text{min}}}{R^*} = K_{HD2} \left( \frac{U \eta_0 \alpha}{E^* R^*} \right)^{0.68} \left( \frac{L}{E^* R^*} \right)^{0.49} \left( \frac{L}{E^* R^*} \right)^{-0.073}
$$
where $K_{HD1}$ and $K_{HD2}$ are constant coefficients of the order of unity. Several other similar approximating formulae have subsequently been produced for various contact configurations. For typical values of the variables involved, values of $h_0 \approx 1$ micrometer are obtained. These compare well with observed values.

4 Boundary Lubrication

It is now known that lubricating layers as small as a single molecule are capable of producing significant improvements in tribological performance (i.e. reduced friction and wear). This is the subject of boundary lubrication.

Even single molecular layers of particular substances attached to solid surfaces can have important effects on tribological behavior. This is the basis of boundary lubrication technology. With the availability of the surface force apparatus (SFA) it is now possible to measure not only the thickness of the lubricating film down to atomic dimensions but also the friction forces involved.

The key to boundary lubrication is the formation and maintenance of a single or multi-molecular layer of lubricating material so as to prevent as much as possible the direct dry contact of the solid surfaces in the tribological couple.

Intervening films such as oxide and sulfide layers have demonstrated to be effective in reducing friction and wear. This is a good example of a chemical film. The oxide material in the film is in intimate contact with the metal surface underneath.

Many other such chemical films are possible and particularly important are those formed when certain organic compounds react with the metal surfaces. Specifically, pure paraffin oil and combinations with small amounts of a fatty acid such as lauric acid, can be very effective in reducing friction. The resulting metallic soap molecules formed at the surface perform well until the temperature becomes high enough that soap melting and film breakdown takes place.

As a general rule, polar molecules exhibit strong affinity for bare metal surfaces and are thus ideal candidates as boundary lubricants. Such molecules may attach to the substrate by physical adsorption, chemical adsorption or by chemical reaction. Most appropriate are straight chain organic molecules with one polar end such as alcohols and soaps of fatty acids.

Often, the presence of more than one molecular layer of lubricating material leads to improved tribological performance. Experimental data shows that some 50 layers of stearic acid deposited onto a stainless steel surface produce a low friction surface over a large number of repeated sliding contacts.

Another way of increasing friction performance with boundary lubrication is to use still single layers but of longer chain molecules. The performance is also improved by the increased stability of the longer chain molecules on the metal surfaces.

A number of intervening solid layers are capable of reducing friction. Surface coatings of materials with layered crystallographic structures, specifically graphite and molybdenum disulphide have been found useful in reducing friction and wear.
5 Summary

In sum, the engineer and the designer have available the following spectrum of unlubricanted and lubricated joint types:

- Solid-solid contact
- Few-atom thick molecular layer lubricants
- Fluid lubricants: Animal fat, vegetable and petroleum-based oils, mineral oils, synthetic oils, and additives
- Greases
- Solid lubricants: Layer- and nonlayer-lattice solids, fullerenes and polymer plastics