1 Introduction

The solutions to typical problems of statics, dynamics, or stability of elastic or anelastic structures are characterized by well-known variational principles. Because these principles assume that the design of the structure is given, they are basically tools of structural analysis. The present paper explores ways in which the principles may be extended to structural design for minimum weight.

To be realistic, a problem of minimum-weight design must involve restrictions on the size of the structural members. Consider, for instance, the minimum-weight design of a straight member of given length for maximum torsional stiffness. The optimal design is a thin-walled cylindrical tube of circular cross section. If there is no restriction on the diameter of the tube, the torsional stiffness can be raised for a given weight by increasing the diameter and reducing the wall thickness until torsional stability becomes a problem. To arrive at a definite optimal design without introducing secondary design considerations, such as torsional stability, one has to restrict the diameter of the tube. The optimal design then places the structural material as close to the surface of the available space as possible.

Minimum-weight design of a cantilever beam for maximum bending stiffness under a tip load can be discussed in a similar manner. To obtain a definite optimal design without invoking considerations of lateral stability, one must restrict the space that is available for the beam, say, to a rectangular prism of given height and breadth. The optimal structure then is a sandwich beam that fully uses this space and places the material in direct stress as close to the relevant faces of the prism as possible.

In the following, only thin-walled or sandwich structures of this kind will be considered. They are characterized by a linear relation between the specific structural stiffness and the specific structural weight. For example, an elastic sandwich beam with a core of the constant height $2H$ and the constant breadth $B$, and identical face sheets of the variable thickness $T < H$, has the specific bending stiffness (ratio of bending moment and curvature) $s = 2E_b HT$ and the specific structural weight $w = 2R_o(a^4 + b^4)$, where $b$ is Young's modulus, $a$ and $b$ are the weights per unit volume of the core and sheet materials, and $T$ is the only design parameter that varies along the beam.

In general, the condition of minimum-weight (optimality condition) involves both the relevant kinematic field (e.g., displacement field of an elastic structure, or velocity field of a rigid, perfectly plastic structure) and the variable specific stiffness. If, however, the specific structural weight is a linear function of the specific stiffness, the optimality condition does not contain the stiffness. This condition then stipulates that a certain specific energy (or power) or the difference between certain specific energies per unit specific stiffness has a constant value throughout the structure. After the displacement field (or velocity field) has been determined from the optimality condition, the variable specific stiffness of the optimal structure may be found from the usual differential equations of the structure.

2 Basic Concepts

The mechanical behavior of an elastic structure at the typical point $x$ of its center line or median surface will be described in terms of generalized strains $q_i$ and the associated generalized stresses $Q_i (i = 1, 2, \ldots, n)$. When the generalized strain $q_i$ (for instance, the principal curvatures of a bent isotropic plate) have been chosen, the associated generalized stresses $Q_i$ (the principal bending moments of the plate) are specified by the requirement that the power of the stresses $Q_i$ on the strain rates $q_i$ be given by $Q_{ij} q_i$, where the repeated letter subscript indicates summation over the range $i = 1, 2, \ldots, n$. In the following, the $n$-dimensional vectors with the rectangular Cartesian components $Q_i$ and $q_i$ will be denoted by $Q$ and $q$, and their scalar product $Q_{ij} q_i$ by $Q q$.

The specific elastic strain energy at the point $x$ of the structure will be written as $w(x)$, where $s = s(x)$ is the specific stiffness at $x$ and $e(x)$ with $q = q(x)$ is the specific strain energy stored in a structural element of unit stiffness that has been subjected to the strain $q$. The function $s(x)$ specifies the design of the structure; the function $e(q)$ does not contain any design parameters.

Since the rate of change of the specific strain energy equals the power of the stresses $Q_i$ on the strain rates $q_i$, we have $Q_{ij} q_i = s(x) q_i q_i = s(x) \|q\|^2$, and hence

$$Q_{ij} = 2s(x) q_i q_i$$

(1)

For the elastic structures considered in this paper, the specific strain energy $w(x)$ is a positive definite quadratic form of the strains $q_i$. Accordingly,

$$Q_{ij} = 2e(x) q_i q_i$$

(2)

by Euler's theorem on homogeneous functions.

In the following, only kinematically admissible strain fields will be considered; that is, fields that are derived from displacement fields satisfying the kinematic boundary conditions of the problem. Because the specific structural weight $w(x)$ is supposed to be a
3 Optimal Elastic Design for Maximum Stiffness

A single load applied at a point \( z \) of an elastic structure and the displacement \( \delta \) produces at this point \( \in \), in general, have distinct directions. If the two directions coincide, the load is said to act in a principal direction at \( z \). As a rule, the principal directions at a point of a structure depend on its design, but symmetry considerations will often enable us to identify a principal direction before the details of the structural design are fixed. In the following, the term "principal direction" will be exclusively used for principal directions of this kind.

At a point \( \delta \) of an elastic structure, let a unit displacement in a principal direction be prescribed. We wish to design a structure of given weight to maximize the intensity \( 2P \) of the load that must be made to act at \( \delta \) in a principal direction to produce the prescribed displacement.

According to the principle of minimum potential energy, the strain field \( g(z) \) that prescribed displacement induces in the elastic structure with the stiffness \( s(z) \) leads to a smaller value of the strain energy

\[
\int \delta(z; \varepsilon) g(z) \, dz
\]

than any other kinematically admissible strain field, and the value of the minimum is \( P \). Maximizing \( P \) by appropriate choice of \( s(z) \) thus amounts to finding

\[
\max_{s(z)} \min_{\varepsilon} \int \delta(z; \varepsilon) g(z) \, dz
\]

for kinematically admissible strain fields \( g(z) \) and for stiffness distributions \( s(z) \) satisfying \( \delta(z; \varepsilon) \) constant.

Let us now consider a one-parameter family of designs \( s(z; \varepsilon) \) that satisfy

\[
\int \delta(z; \varepsilon) \, dz = \text{Const.}
\]

the optimal design being \( s(z; 0) \). For the design \( s(z; \varepsilon) \), let the minimum strain energy be furnished by the kinematically admissible strain field \( g(z) \). Note that the load \( 2P \) that must be applied to this design to produce the prescribed displacement is found from

\[
P(\varepsilon, \gamma) = \int \delta(z; \varepsilon) g(z; \gamma) \, dz
\]

The relation (9) suggests the introduction of the expression

\[
P(\varepsilon, \gamma) = \frac{1}{2} \int \delta(z; \varepsilon) g(z; \gamma) \, dz
\]

For a fixed design \( s(z; \varepsilon) \), it follows from the principle of minimum potential energy that \( P(\varepsilon, \gamma) \geq P(\varepsilon, \varepsilon) \). Equality applies only for \( \gamma = \varepsilon \). In particular, \( P(0, \varepsilon) \geq P(0, 0) \) with equality for \( \varepsilon = 0 \) only. On the other hand, it follows from the optimality of the design \( s(z; 0) \) that \( P(0, 0) \geq P(\varepsilon, \varepsilon) \). Thus

\[
P(0, \varepsilon) \geq P(0, 0) \geq P(\varepsilon, \varepsilon)
\]

and hence

\[
(\delta P/\delta \varepsilon)_{\varepsilon = 0} = 0, \quad (\delta P/\delta \varepsilon)_{\varepsilon = 0} = 0,
\]

where the subscript \( 0 \) indicates \( \varepsilon = \gamma = 0 \).

Note that the first equation (9) must yield the differential equations and natural (static) boundary conditions for the displacement field of the optimal structure. Since these are known in advance, there is no need of obtaining them by variational methods. On the other hand, \( \delta(z; \varepsilon) / \delta \varepsilon \) in the second equation (9) cannot be chosen arbitrarily but must have vanishing integral over the structure. Combining this condition with the one following from the second equation (9), we write

\[
\int \left( \frac{\partial \delta(z; \varepsilon)}{\partial \varepsilon} \right) g(z; \varepsilon) \, dz = 0,
\]

where \( C \) is a constant and \( \delta(z; \varepsilon) / \delta \varepsilon \) can now be treated as an arbitrary function of \( \varepsilon \). We thus have the optimality condition

\[
e\varepsilon = 0 = C/2
\]

The strains that the prescribed displacement induces in the optimal structure correspond to a constant specific strain energy per unit specific stiffness. When kinematically admissible strains of this type have been found, the variation of the stiffness \( s(z) \) over the structure can be obtained from the differential equations and static boundary conditions for the displacement field of the structure.

Note that \( e(\varepsilon; 0) \) may be regarded as a scalar measure of the strain intensity at the point \( z \) of the structure. The optimality condition (11) may then be expressed by stating that the optimal structure is uniformly strained.

Proof of Optimality. By the principle of minimum potential energy,

\[
P(\varepsilon, 0) \geq P(\varepsilon, \varepsilon).
\]

Setting

\[
\delta(z; \varepsilon) = \delta(z; 0) + \delta(z; \varepsilon),
\]

we have

\[
\int \delta(z; \varepsilon) g(z) \, dz = 0
\]

because \( \int \delta(z; \varepsilon) g(z) \, dz \) has a constant value for the considered design. Substitution of (13) into the definition (7) for \( P(\varepsilon, 0) \) and use of the optimality condition (11) and the relation (14) furnishes

\[
P(\varepsilon, 0) = P(0, 0) + \int \delta(z; \varepsilon) e(\varepsilon; 0) g(z) \, dz = P(0, 0).
\]

The inequality (12) thus is equivalent to

\[
P(0, 0) \geq P(\varepsilon, \varepsilon)
\]

Since this relation is valid for any value of \( \varepsilon \) and independently of the choice of the one-parameter family of designs of constant weight into which the design \( e(\varepsilon; 0) \) has been embedded, we have established absolute optimality.

The condition (11) therefore is not only necessary but also sufficient for optimality.

Example. Consider a sandwich beam of the kind described in Section 2. This beam is to be built in at \( z = -l \) and to be designed for maximum stiffness with respect to a transverse load \( 2P \) applied at \( z = z_o \).

The specific strain energy of a sandwich beam per unit specific stiffness is given by one half of the square of the curvature. If the deflection is denoted by \( u(z) \), the optimality condition (11) requires that

\[
u' = \pm C
\]

A continuously differentiable function \( u(z) \) that satisfies (17) and the boundary conditions \( u(\pm l) = u'(\pm l) = 0 \) is readily seen to have

\[
u'(z) = \begin{cases} -C & \text{for } 0 \leq |z| < l/2, \\ C & \text{for } 1/2 < |z| \leq l, \end{cases}
\]

and

\[
u(0) = CP/4
\]

Now, curvature and bending moment have the same sign; the

The necessity of this condition for maximum stiffness of an elastic structure of given weight has been stressed in a series of papers by Wszelényi (see [1] for an exhaustive list of references), but the proofs of sufficiency and absolute optimality given subsequently appear to be new.

Numbers in brackets designate References at end of paper.
bending moment $M(x)$ must therefore change sign at $x = \pm 1/2$. This condition removes the indeterminacy of the bending moment. For example, for $x_0 = 1/4$ one finds

$$M(x) = \begin{cases} \frac{Pl}{4} \left( 1 + \frac{z}{l} \right) & \text{for } -l \leq x \leq l/4, \\ \frac{Pl}{4} \left( 3 - \frac{z}{l} \right) & \text{for } l/4 \leq x \leq l. \end{cases}$$  \hspace{1cm} (20)

The specific stiffness $s(x)$ equals the quotient of the absolute value of the bending moment by the absolute value $C$ of the curvature. It therefore follows from (19) and (20) that

$$s(x) = \begin{cases} \frac{PP}{16a(0)} \left( 1 + \frac{z}{l} \right) & \text{for } -l \leq x \leq l/4, \\ \frac{PP}{16a(0)} \left( 3 - \frac{z}{l} \right) & \text{for } l/4 \leq x \leq l. \end{cases}$$  \hspace{1cm} (21)

From (21), the average specific stiffness of the optimal design is found to be

$$s_{av} = \frac{7PP}{128a(0)}$$  \hspace{1cm} (22)

To assume the same central deflection $u(0)$ under the same load $2P$ at $x = 1/4$, a beam of constant cross section must be given the specific stiffness

$$s = \frac{PP}{128a(0)},$$  \hspace{1cm} (23)

which exceeds the average specific stiffness of the optimal design by 28.6 percent.

Note that, for $x_0 > 1/2$, it follows from the condition of vanishing bending moment at $x = \pm 1/2$ that the bending moment vanishes identically in $-1 \leq x \leq x_0$. The optimal design is a cantilever beam with the free end at $x = x_0$ and the built-in end at $x = 1$. A similar remark applies when $x_0 < -1/2$.

4 Optimal Elastic Design for Maximum Fundamental Frequency

This section is concerned with an elastic structure of given weight that is to carry given masses at specific points and is to be designed to maximize the fundamental frequency. In view of the assumed linear relationship between unit weight and stiffness, we write the specific structural mass as $a^2 + b^2u(x)$, where $a$ and $b$ are constants and $s(x)$ is the variable stiffness.

Consider the time-dependent displacement field $u(x, t)$, where $u(x)$ satisfies the kinematic boundary conditions and $s(x)$ is the lowest natural frequency. The strain field associated with the displacement field $u(x, t)$ will be denoted by $q(x)$. According to Rayleigh's principle, the square of the fundamental frequency $\omega$ is the minimum of a certain quotient. Maximizing $\omega$ by an appropriate choice of $s(x)$ thus amounts to finding

$$\max_{s(x)} \left\{ \min_{\omega(s)} \left[ \frac{\int q(x) \cdot q(x) dx}{\frac{1}{2} \int \left[ a^2 + b^2u(x) \right] \left[ u(x) \right]^2 dx + \frac{1}{2} \sum M_f [u(x)]^2 dx \right] \right] \right\},$$  \hspace{1cm} (24)

where the sum in the denominator is extended over all point masses $M_f$, the amplitude of $M_f$ being $u(x)$.

As in Section 3, we consider a one-parameter family of designs $s(x; \eta)$ that satisfy (5), the design with the greatest fundamental frequency $\omega$ being $s(x; 0)$. The fundamental mode of the design $s(x; \eta)$ will be denoted by $u(x; \eta)$ and the corresponding strain field by $q(x; \eta)$. In analogy to (7), we introduce the expression

$$P(\eta, \tau) = \frac{1}{2} \int \left[ a^2 + b^2u(x; \eta) \right] \left[ u(x; \eta) \right]^2 dx + \frac{1}{2} \sum M_f [u(x; \eta)]^2 dx.$$

It will be convenient to denote numerator and denominator in (25) by $E(\epsilon, \eta)$ and $K(\epsilon, \eta)$, respectively. Using Rayleigh's principle and the optimality of the design $s(x; 0)$, we again obtain (8) and hence (9). The first condition (9) furnishes the differential equations and natural boundary conditions of the fundamental mode of the optimal design, while the second condition (9) yields, after division by

$$\frac{1}{2} \int \left[ a^2 + b^2u(x; 0) \right] \left[ u(x; 0) \right]^2 dx + \frac{1}{2} \sum M_f [u(x; 0)]^2 dx$$

and use of Rayleigh's principle,

$$\int \left[ \frac{\partial q(x; \eta)}{\partial \eta} \right] \left\{ q(x; 0) \right\} dx = 0$$  \hspace{1cm} (26)

Using (5) as in (10), we finally obtain the optimality condition

$$q(x; 0) - \frac{b^2}{2} \omega^2 [u(x; 0)]^2 = C \eta^2$$  \hspace{1cm} (27)

The right-hand side of (27) has been written as $C \eta^2/2$ for the following reason. The structure will be supported at some point $x$ where $u(x; 0) = 0$. At this point, the left side of (27) reduces to the specific strain energy $E(\epsilon, \eta)$, which is nonnegative.

The optimality condition (27) states that, for the fundamental mode, the difference between the amplitude of the specific strain energy per unit stiffness and the amplitude of the specific kinetic energy of the structure (exclusive of the point masses) per unit stiffness is constant over the structure.

In view of its derivation, the condition (27) is necessary for optimality; that it also is sufficient may be seen as follows. The inequality (12) here is an immediate consequence of Rayleigh's principle. With $E(\epsilon, \eta)$ and $K(\epsilon, \eta)$ as defined in connection with (25), we have

$$P(\eta, 0) = E(\epsilon, 0) = E(0, 0) + \frac{1}{2} \int q(x; \eta) \cdot q(x; 0) dx,$$

$$P(0, 0) = K(\epsilon, 0) = \int \left[ a^2 + b^2u(x; 0) \right] \left[ u(x; 0) \right]^2 dx,$$

where (13) has been used. We now subtract (29) from (28), take account of the fact that $P(0, 0) = \omega^2$, and use (27) and (14); thus

$$P(\epsilon, 0) - P(0, 0) = K(\epsilon, 0) = 0$$  \hspace{1cm} (30)

Since $K(\epsilon, 0) > 0$ by definition, $P(\epsilon, 0) = P(0, 0)$ and, hence, $P(0, 0) = K(\epsilon, 0)$ by (12). The condition (27) is then seen to be sufficient as well as necessary for optimality, and we have established again absolute optimality.

Example. A vertical elastic rod of the variable cross-sectional area $A(x)$ is fixed at the upper end $x = 0$ and carries a given mass $M$ at the lower end $x = 1$. The rod is to be optimally designed for maximum fundamental frequency of longitudinal vibrations. Here, specific stiffness and mass are $Ea$ and $\rho a$, where $E$ and $\rho$ denote Young's modulus and density. Accordingly $a^2 = 0$, $b^2 = 1/\rho$, and $\omega$ is the speed of propagation of longitudinal waves in the rod. The specific strain energy per unit stiffness is one half of the square of the longitudinal strain. If the longitudinal displacement is denoted by $u(x)$, the optimality condition (27) therefore requires that

$$u^2 - \frac{a^2}{\omega^2} u^2 = C \eta^2$$  \hspace{1cm} (31)

With the initial condition $u(0) = 0$, the differential equation (31) yields

$$u(x) = c \sinh \frac{\omega x}{c}$$  \hspace{1cm} (32)
Substituting (32) into the differential equation for the fundamental mode, namely,

$$[s(x)u''(x)]' + \frac{\omega^2}{c^2} s(x)u(x) = 0,$$  \hspace{1cm} (33)

and solving for \( s(x) \), one finds

$$s(x) = \frac{\text{const}}{\cosh^3(\omega x / c)}.$$  \hspace{1cm} (34)

The value of the constant in (34) is obtained from the boundary condition \( s(0)u'(0) = (\mu u) \). Thus

$$s(x) = \frac{\cos M \sinh (\omega x / c) \cosh (\omega l / c)}{\cosh^3(\omega x / c)}.$$  \hspace{1cm} (35)

This result has been obtained in different manners by Turner [2] and Taylor [3]. Niordson [4] has treated the analogous problem for a vibrating cantilever beam with a solid cross section of constant shape but variable size. As has been pointed out in Section 1, the optimal design places the structural material as close as is possible to the surface of the space available for the structure. Solid sections thus are less advantageous than the sandwich sections considered here. Moreover, by restricting the present analysis to the linear relationship between specific structural weight and stiffness that is typical for sandwich structures, we achieve a decomposition of the solution of the problem into the integration of a nonlinear optimality condition for the displacement field, which does not involve the specific stiffness, and the subsequent determination of this stiffness from the differential equation for the already known displacement field.

5 Optimum Elastic Design for Maximum Buckling Load

This section is concerned with an elastic structure under a load that brings the structure to a bifurcation point of elastic equilibrium. The intensity of this buckling load is given by the minimum of a Rayleigh quotient. Both numerator and denominator of this quotient are related to the transition from the unbuckled to an adjacent buckled configuration. Specifically, the numerator represents the strain energy associated with this transition, while the denominator represents the work per unit load intensity that the buckling load does in this transition. The problem of minimizing this Rayleigh quotient with respect to the displacement field and then maximizing the minimum with respect to the stiffness can be stated in exactly the same manner as in Section 4. In contrast to the quotient in (24), however, the present Rayleigh quotient does not contain the stiffness in the denominator. The resulting optimality condition is, therefore, less involved than (27); in fact, it has the form (11). As in Section 4, it can be shown to be sufficient for absolute optimality.

Example. A sandwich column is pin-supported at the ends \( z = 0 \) and \( z = 2L \). It is to have minimum weight for a given buckling load \( P \).

If \( u(x) \) denotes the lateral displacement from the fully loaded but unbuckled configuration of the column, the buckling load \( P \) is the minimum value of the quotient

$$\frac{\int [s(x)u'(x)]^2 dx}{\int [u(x)]^2 dx}.$$  \hspace{1cm} (36)

over all displacement fields \( u(x) \in C^1 \) that satisfy the conditions \( u(0) = u(2L) = 0 \) and have no zeros in \( 0 < c < 2L \). The denominator in (36) represents the amount by which the ends of the column approach each other as a result of the deflection \( u(x) \).

The optimality condition (11) here reduces to

$$u'(x) = \pm C.$$  \hspace{1cm} (37)

Since the bending moment \( M(x) \) in the buckled state satisfies

$$M(x) = Pu(x) = -s(x)u''(x),$$  \hspace{1cm} (38)

the lower sign must be taken in (37). In view of the boundary conditions, we thus have

$$u(x) = \frac{C}{2} x(2L - x)$$  \hspace{1cm} (39)

and, by (38),

$$s(x) = \frac{P}{2} x(2L - x).$$  \hspace{1cm} (40)

Similar problems of optimal design of columns have been treated in other ways by Keller [5], Tadjbakhsh and Keller [6], Keller and Niordson [7], and Taylor [8] without taking account of the fact that the optimum design is of sandwich type, with a linear relationship between specific structural weight and stiffness. The remarks made at the end of Section 4 apply to these buckling problems as well.

Note that the specific bending stiffness \( s(x) \) given by (40) and, hence, the thickness of the face sheets of the sandwich section vanish at the ends of the column. To avoid this unrealistic feature of the design (40), we may prescribe a minimum value \( s_0 \) below which the specific bending stiffness is not allowed to drop. The optimality condition (37), with the minus sign, will then only apply in some central interval \( \xi \leq z \leq 2L - \xi \), while in the remainder of the length of the column the specific bending stiffness will have the constant value \( s_0 \). Thus

$$M(x) = Pu(x) = -au''(x)$$ \hspace{1cm} for \( 0 \leq x \leq \xi \)  \hspace{1cm} (41)

With the initial condition \( u(0) = 0 \), this differential equation yields

$$u(x) = A \sin \omega x \quad \text{for} \quad 0 \leq x \leq \xi,$$  \hspace{1cm} (42)

where \( A \) is a constant of integration and \( \omega = (P/s_0)^{1/4} \). On the other hand, the optimality condition (37), the continuity of the deflection \( u \) at \( x = \xi \), and the symmetry condition \( u'(0) = 0 \) furnish

$$u(x) = A \sin \omega x + \frac{1}{2} C(x - \xi)(2L - x - \xi)$$ \hspace{1cm} for \( \xi \leq x \leq 2L - \xi \).  \hspace{1cm} (43)

The constants \( A \) and \( C \) in (42) and (43) can be eliminated by the use of the continuity of \( u' \) and \( u'' \) at \( x = \xi \). Thus

$$\omega \xi + \cot \omega \xi = C$$  \hspace{1cm} (44)

The lowest zero of this transcendental equation in \( \omega \xi \) furnishes the value of \( \xi \) when the buckling load \( P \) and the minimum stiffness \( s_0 \) and, hence, \( \omega = (P/s_0)^{1/4} \), are given.

6 Optimal Plastic Design for Maximum Safety

In this section, which concerns the safety of a rigid, perfectly plastic structure against collapse by plastic flow under constant loads, the symbols \( u(x), q(x), s(x), \) and \( e(x) \) will be used to denote the velocity field of this plastic flow, the corresponding plastic strain rate field, the specific plastic resistance, and the specific rate of dissipation per unit plastic resistance. Note that \( e(x) \) is convex and positive homogeneous of deg 1 in the strain rates \( q_1, q_2, \ldots, q_n \) (see, for instance, [9, p. 18]) and free from design parameters. For an arbitrary positive scalar \( \lambda \) and strain rates \( q \) and \( g \), we thus have

$$\varepsilon(0) = 0,$$

$$\varepsilon(\lambda q) = \lambda \varepsilon(q), \quad \lambda > 0,$$

$$\varepsilon(q + g) \leq \varepsilon(q) + \varepsilon(g).$$  \hspace{1cm} (45)

In a stress space with the rectangular Cartesian coordinates \( Q_1, Q_2, \ldots, Q_n \) of stress that are attainable at the point \( x \) of the structure are represented by points of a convex domain
called the attainable domain. The function \( s(x) = q(x) \) is the supporting function of this domain. A given plastic strain rate \( q_0, q_1, \ldots, q_n \) can be produced by any state of stress \( Q_0, Q_1, \ldots, Q_n \) that is represented by the intersection of the attainable domain with its supporting plane for the direction \( q \). Where \( e(q) \) has continuous partial derivatives \( \partial e/\partial q \), the state of stress producing the plastic strain rate \( q \) is uniquely determined by \( (1) \).

For the rigid, perfectly plastic sandwich structures considered here, the specific structural weight is a linear function of the specific plastic resistance.

The problem of maximizing the safety of a rigid, perfectly plastic structure of given weight amounts to finding

\[
\max_{q(x)} \left\{ \min_{q(x)} \left\{ \int p(x)q(x)dx \right\} \right\} \tag{46}
\]

Here, \( p(x)u(x) \) is the specific power of the loads \( p(x) \) on the kinematically admissible velocities \( u(x) \).

The problem \( (46) \) can obviously be treated in the same way as the problem \( (24) \). The optimality condition again has the form \( (11) \), with the new meaning of \( e(q(x)) \); it insures absolute optimality. This optimality condition has been derived in a different way by Drucker and Shield [10]; it has been applied to numerous problems (for surveys of the literature, see [11], [12]).

Example. A rigid, perfectly plastic sandwich plate is simply supported along its strictly convex edge, which has continuously turning tangent. The plate is to carry a uniformly distributed load of the intensity \( F \) (per unit area), and its face sheets are to obey Tresca's yield condition. The plate is to be designed to maximize safety for a given structural weight.

If \( q_1 \) and \( q_2 \) denote the principal rates of curvature of the plate, the specific rate of dissipation is given by

\[
q_1 q_2 > 0 \tag{48}
\]

and

\[
e(q) = 2\sigma HT(q_1 + q_2) \tag{49}
\]

We may therefore set

\[
q = 4\sigma HT, \quad e(q) = (q_1 + q_2)/2 \tag{50}
\]

Introducing rectangular Cartesian coordinates \( x, y \) in the midplane of the plate, we denote the rate of deflection by \( u(x, y) \). The rates of curvature in the coordinate directions then are \(-\partial^2 u/\partial x^2 - \partial^2 u/\partial y^2 \), and the rate of twist for the coordinate directions is \(-\partial^2 u/\partial x \partial y \). In terms of these rates, we have

\[
q_1 + q_2 = -\left( \partial^2 u/\partial x^2 \right) - \left( \partial^2 u/\partial y^2 \right), \tag{51}
\]

and

\[
q_1 q_2 = -\left( \partial^2 u/\partial x \partial y \right) - \left( \partial^2 u/\partial x \partial y \right). \tag{52}
\]

In view of \( (50) \) and \( (51) \), the optimality condition \( (11) \) requires that

\[
\left( \partial^2 u/\partial x^2 \right) + \left( \partial^2 u/\partial y^2 \right) = -C. \tag{53}
\]

Together with the condition of vanishing deflection along the edge of the plate, the differential equation \( (53) \) uniquely determines the rate of deflections \( u(x, y) \) of the optimal design. If this rate satisfies \( (48) \) with \( (52) \), the assumption regarding the sign of the principal curvatures turns out to be valid. We shall assume that this is the case.

Application of \( (1) \) to \( (49) \) furnishes the principal bending moments

\[
M_1 = M_2 = 2\sigma HT, \tag{54}
\]

where \( T = T(x, y) \). Since the principal bending moments are equal to each other, the bending moments \( M_1, M_2 \) in the coordinate directions have the same value \( 2\sigma HT \), and the twisting moment \( M_{xy} \) vanishes. The equilibrium condition,

\[
\left( \partial^2 M_1/\partial x^2 \right) + 2\left( \partial^2 M_1/\partial x \partial y \right) + \left( \partial^2 M_2/\partial y^2 \right) = -\varphi, \tag{55}
\]

in which \( \varphi \) denotes the safety factor, therefore furnishes

\[
\left( \partial^2 T/\partial x^2 \right) + \left( \partial^2 T/\partial y^2 \right) = -\varphi p/(2\sigma H). \tag{56}
\]

Together with the condition that the bending moment normal to the edge, and hence the thickness \( T \), vanish along the edge, the differential equation \( (56) \) uniquely determines the thickness \( T(x, y) \) of the optimal plate that has the given safety factor \( \varphi \).

References


