DYNAMIC ELASTIC SOLUTIONS IN NEO-HOOKEAN AND MOONEY-RIVLIN MATERIALS

C. E. MANESCHY,* M. MASSOUDI† and V. R. VELLOSO‡

*Department of Mechanical Engineering, University of Pittsburgh, Pittsburgh, PA 15261, U.S.A.
†Department of Energy, Pittsburgh Energy Technology Center, Pittsburgh, PA 15236-0940, U.S.A.
‡Department of Mechanical Engineering, FUNREI, Sao Joao del Rei, MG, Brazil

(Received 31 September 1992; accepted 1 February 1993)

Abstract—Inhomogeneous motions are discussed within the context of non-linear constitutive theories. Deformations set up by time-dependent applied forces or displacements are analyzed for an infinite slab of neo-Hookean material and a circular cylinder of a Mooney-Rivlin type. The problems reduce to obtaining the solutions for simple linear differential equations.

1. INTRODUCTION

Recently, there has been a great deal of interest in the study of inhomogeneous deformations of non-linearly elastic materials. According to Ericksen's result [1, 2] such deformations are possible only within the context of specific non-linear theories. That is, if we are interested in investigating a prescribed inhomogeneous motion, we would have to study it in the context of some subclass of isotropic elastic materials. This has been the approach used to find the solutions to many problems concerning this kind of motion [3–5]. All these works have basically dealt with time-independent deformations.

In regard to dynamic deformations, it can be said that only a few exact solutions have been found for non-linearly elastic materials. Moreover, most of these solutions were based on general theories of incompressible and compressible materials (the so-called universal solutions [6, 7]). Since, in general, an inhomogeneous deformation can only be sustained by certain types of solids, emphasis should be put on establishing what class of material is capable of satisfying all the field and boundary conditions for a given deformation (cf. ref. [8]).

This work aims at discussing two different sets of problems focusing on specific constitutive theory applied to time-dependent deformations. First, the solution to the problem of an infinite slab of neo-Hookean material subject to shear deformation is obtained. Second, we study the problem of a cylinder under shear motion within the context of the Mooney-Rivlin theory.

2. DYNAMIC SOLUTIONS IN NEO-HOOKEAN AND MOONEY-RIVLIN MATERIALS

As the first example, we will study the time-dependent shearing deformation of an elastic layer. If \( X, Y, Z \) denote the undeformed coordinates of a particle and \( x, y, z \) the coordinates of the same particle in its deformed configuration, this deformation is given by

\[
\begin{align*}
  x &= Xf(Z, t), \\
  y &= Y, \\
  z &= Zg(Z, t).
\end{align*}
\]


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*Permanent address: Department of Mechanical Engineering, Pontifícia Universidade Católica-RJ, Rio de Janeiro, RJ, Brazil.
†Contributed by K. R. Rajagopal.
The deformation gradient associated with this motion is
\[
F = \begin{pmatrix}
  f & 0 & Xf' \\
  0 & 1 & 0 \\
  0 & 0 & 1/f
\end{pmatrix},
\]
(2.4)
where the prime denotes differentiation with respect to \( Z \). In arriving at equation (2.4), the incompressibility condition
\[
f = \frac{1}{Zg'},
\]
(2.5)
was used. The Cauchy–Green strain tensor, \( B = FF^T \), is
\[
B = \begin{pmatrix}
  f^2 + (Xf')^2 & 0 & Xf'/f \\
  0 & 1 & 0 \\
  Xf'/f & 0 & 1/f^2
\end{pmatrix}.
\]
(2.6)
For a neo-Hookean material, the Cauchy stress is given by
\[
T = -pI + \mu B,
\]
(2.7)
where the shear modulus \( \mu \) is a constant and the term \(-pI\) is the indeterminate part of the stress due to the constraint of incompressibility.

The equation of motion, in the absence of body forces, is
\[
\text{div } T = \rho \frac{\partial^2 X}{\partial t^2} \quad \text{or} \quad \frac{\partial T_{ij}}{\partial X_p} \frac{\partial X_p}{\partial X_j} = \frac{\partial^2 X_i}{\partial t^2}.
\]
(2.8)
Recognizing that \( \dot{X}_p / \dot{X}_j = (F^{-1})_{ij} \) and substituting equation (2.7) into equation (2.8), we obtain
\[
\frac{\partial p}{\partial X} = \mu \dot{X} X_i \dot{X}_j \quad \text{or} \quad \frac{\partial p}{\partial X} = \mu \frac{f'}{f} X_i \frac{\partial p}{\partial X} - \rho X_i.
\]
(2.9)
where \( p = p(X, Z) \) and the dot denotes differentiation with respect to time. By cross-differentiating the equations above and eliminating the pressure term, we find
\[
\mu (f'' - f''') = \rho (\dot{f}' - \ddot{f}'').
\]
(2.10)
The solution to equation (2.11) is found by separation of variables to be
\[
f(Z, t) = (C_1 e^{C_2t} + C_3 e^{-C_2t}) \phi(t),
\]
(2.12)
where \( C_1, C_2, C_3 \) are constants and \( \phi(t) \) is an arbitrary function of time. The interesting feature displayed by this solution is that it allows the deformation defined in equations (2.1)–(2.3) to be sustained by a neo-Hookean material regardless of the form for the time-dependent function \( \phi(t) \). This gives some flexibility in choosing the class of function \( f(Z, t) \) that will satisfy the boundary conditions. For example, if the layer at \( Z = 0 \) has the motion
\[
x(0, t) = AX \cos \omega t,
\]
(2.13)
the solution for \( f(Z, t) \) can be of the form
\[
f(Z, t) = Re^\{i(C_1 e^{C_2t} + C_3 e^{-C_2t})e^{-i\omega t}\}.
\]
(2.14)
Equation (2.14) allows the function \( g(Z, t) \) to have the oscillatory structure
\[
g(Z, t) = Re^\{i\phi(Z)e^{-i\omega t}\},
\]
(2.15)
where \( \phi(Z) \) is found from the incompressibility condition (2.5) to be
\[
\phi(Z) = \frac{1}{ZC_3 \sqrt{C_1 C_2}} \tan^{-1} \left( \frac{\sqrt{C_1}}{\sqrt{C_2}} e^{i\omega t} \right) - \tan^{-1} \left( \frac{\sqrt{C_2}}{\sqrt{C_1}} \right).
\]
(2.16)
Let us turn our attention to the determination of the set of boundary conditions needed to evaluate all the constants. From equations (2.1) and (2.13)

\[ f(0, t) = A \cos \omega t. \]  
\[ \text{(2.17)} \]

If the layer \( Z = H \) is assumed to be stress-free, it follows from equations (2.6) and (2.7) that

\[ p(X, H, t) = \frac{\mu}{f^3(H, t)}, \]  
\[ f'(H, t) = 0. \]  
\[ \text{(2.18)} \] \[ \text{(2.19)} \]

Integrating equations (2.9) and (2.10), we arrive at

\[ p(X, Z, t) = \frac{X^2}{2} f(\mu f'' - \rho f') - \int \left( \frac{\mu f'}{f^3} + \frac{\rho Z}{f} \right) dZ + H(t). \]  
\[ \text{(2.20)} \]

Equation (2.20) when evaluated at \( Z = H \) can be compared with equation (2.18) to yield

\[ \mu f''(H, t) = \rho f'(H, t). \]  
\[ \text{(2.21)} \]

The constants are then found by using equations (2.17), (2.19) and (2.21), and the analytical solution to the problem is completely determined.

Assume, as the second example of an elastodynamic problem, the shearing deformation of a circular cylinder. This motion is prescribed by the mapping

\[ r = R, \quad 0 \leq R \leq R_0, \]  
\[ \theta = \Theta, \]  
\[ z = Z + \phi(R, t), \]  
\[ \text{(2.22)} \] \[ \text{(2.23)} \] \[ \text{(2.24)} \]

and it was discussed by Rajagopal et al. [3] for neo-Hookean material. If a Mooney-Rivlin type material, for which the stress is

\[ \mathbf{T} = -p \mathbf{I} + \mu \left( \beta + \frac{1}{2} \right) \mathbf{B} - \mu \left( \frac{1}{2} - \beta \right) \mathbf{B}^{-1}; \quad -\frac{1}{2} \leq \beta \leq \frac{1}{2}, \]  
\[ \text{(2.25)} \]

is here considered, it can be shown that equation of motion (2.8) yields the following expressions

\[ \frac{\partial p(R, t)}{\partial R} = -\mu \left( \frac{1}{2} - \beta \right) \frac{1}{R} \frac{\partial}{\partial R} \left[ R \left( \frac{\partial \phi}{\partial R} \right)^2 \right], \]  
\[ \text{(2.26)} \]

\[ \frac{\partial p(R, t)}{\partial Z} = \mu \frac{\partial}{\partial R} \left( R \frac{\partial \phi}{\partial R} \right) - \rho \frac{\partial^2 \phi}{\partial t^2}. \]  
\[ \text{(2.27)} \]

Through the change of variable

\[ \tilde{p}(R, t) = p(R, t) \mu \left( \frac{1}{2} - \beta \right) \left[ \left( \frac{\partial \phi}{\partial R} \right)^2 + \int \left( \frac{\partial \phi}{\partial R} \right)^2 dR \right], \]

equations (2.26) and (2.27) will reduce to

\[ \frac{\mu}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \phi}{\partial R} \right) - \rho \frac{\partial^2 \phi}{\partial t^2} = F(t), \]  
\[ \text{(2.28)} \]

where \( F(t) \) is an arbitrary function. Let us assume that at the outside surface, \( R = R_0 \), there is applied a stress of the form

\[ \tau_{rz} = A + Be^{-\lambda t}; \quad \lambda > 0. \]  
\[ \text{(2.29)} \]

From the expression (2.25) we find the shear stress \( \tau_{rz} \) to be given by

\[ \tau_{rz} = \mu \frac{\partial \phi}{\partial R}. \]  
\[ \text{(2.30)} \]
Comparing equations (2.29) with (2.30) we require the function $\phi(r, t)$ to have an exponential-type structure. Hence, equation (2.28) can be rewritten as

$$\frac{\mu}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \phi}{\partial R} \right) - \rho \frac{\partial^2 \phi}{\partial t^2} = C_1 + C_2 e^{-at},$$

which allows solution of the form

$$\phi(R, t) = \frac{C_1}{4\mu} R^2 + \left[ \frac{C_2}{\lambda^2} + C_3 I_0(xR) \right] e^{-at} + C_4; \quad x^2 = \frac{\rho \lambda^2}{\mu} > 0,$$

where $I_0(xR)$ is a modified Bessel function and $C_1, C_2, C_3, C_4$ are constants that can be found so as to satisfy equation (2.29). Therefore, from equations (2.30) and (2.29)

$$C_1 = \frac{2A}{R_0^2}, \quad C_3 = -\frac{B}{\sqrt{\rho \mu \lambda I_0(xR_0)}}$$

and

$$\frac{C_2}{\lambda^2} e^{-at} + C_4 = H(t),$$

which represents rigid body motion and can be assumed to vanish without loss of generality.

3. CONCLUSIONS

Inhomogeneous deformations and motions of non-linearly elastic solids have been studied in this paper. The aim of the work was to set up analytical solutions for dynamic problems under the context of neo-Hookean and Mooney-Rivlin theories. It was observed that generalizations of some static deformations allow an interesting class of unsteady solutions. That was the case in the first example discussed in section 2, where standing wave solutions have been found. Also, it was shown that shear dynamic deformations in cylinders are possible in a more general class of material such as the Mooney-Rivlin.

Finally, it should be pointed out that the deformations investigated in this work are just a few of many motion problems for which analytical solutions can be determined. These solutions can be useful in providing a check for the numerical schemes used in more complex problems.

REFERENCES