In most problems of stress analysis, the stress field is found within a solid body without regard for local effects caused by the application of the load; however, when two solid bodies with curved surfaces are forced together, consideration must be given to the special stress field created near the contact area. Gears and rolling-element bearings are two notable examples of machine parts in which contact stresses are of great importance in determining the operating life.

Contact problems are classified as counterformal if the dimensions of the area of contact are small compared to the radii of curvature of the contacting surfaces near the region of contact. If the dimensions of the contact area are not small with respect to the radii of curvature of the contacting surfaces, the problem is classified as conformal. A counterformal problem is called Hertzian if the contacting surfaces can be approximated by quadratic functions in the region of contact. If the quadratic approximation is invalid, the problem is non-Hertzian. All conformal problems are non-Hertzian.

The following discussion presents an outline of the analysis of Hertzian contact stresses when two bodies with arbitrarily curved surfaces are pressed together. Charts
and figures are included for use in solving various Hertzian problems. Rolling contact problems, contact stresses with friction, and the fatigue behavior of bodies subjected to repeated applications of contact loading are briefly described.

### 9.1 NOTATION

The units for most of the definitions are given in parentheses, using \( L \) for length and \( F \) for force.

- \( a \) Semimajor axis of contact ellipse \((L)\)
- \( A, B \) Coefficients in equation for locus of contacting points initially separated by the same distance \((L^{-1})\)
- \( A_c \) Contact area \((L^2)\)
- \( b \) Semiminor axis of contact ellipse \((L)\)
- \( d \) Rigid distance of approach of contacting bodies \((L)\); also total elastic deformation at origin
- \( E \) Modulus of elasticity \((F/L^2)\)
- \( f \) Friction coefficient
- \( F \) Force \((F)\)
- \( k \) Ratio of major to minor axis of contact ellipse, \( = b/a \)
- \( p \) Pressure \((F/L^2)\)
- \( q \) Line-distributed load \((F/L)\)
- \( R \) and \( R' \) Minimum and maximum radii of curvature for contacting surfaces \((L)\)
- \( z_s \) Distance below center of contact ellipse where maximum shear stress occurs \((L)\)
- \( \theta \) Angle between planes containing principal radii of curvature for contacting bodies
- \( \nu \) Poisson’s ratio
- \( \sigma_c \) Maximum compressive stress \((F/L^2)\)
- \( \sigma_{ys} \) Yield strength in tension \((F/L^2)\)
- \( \tau_{\text{max}} \) Maximum shear stress \((F/L^2)\)
- \( 1, 2 \) Subscripts designating bodies 1 and 2

#### Geometric Characteristics of Surfaces

Consider a surface \( F(x, y, z) = 0 \) (Fig. 9-1). At any point on the surface, the normal to the surface is \( \text{grad}(F) \). Let a plane pass through the length of the surface normal at point \( O \), creating a normal section. The intersection of this plane with the surface is a curve in the normal section of the surface at point \( O \). An infinite number of normal sections may be taken through any point on the surface. The following theorem holds [9.1]: At any point of a surface, two normal sections exist for which the radii of curvature are a minimum and a maximum; the planes that each contain one of these normal sections are
perpendicular. The following terminology is adopted here: the normal sections that have either a minimum or a maximum radius of curvature are called the \textit{principal normal sections} of the surface at the point. The minimum and maximum radii of curvature are called the \textit{principal radii of curvature of normal sections} at the point. The tangents to the curvatures in the principal normal sections at point \(O\) are called the \textit{principal directions}, and the planes that create the principal normal sections are called the \textit{principal planes} of curvature. Equations for computing the principal radii of curvature and principal directions at a point of a surface are presented in any treatise dealing with differential geometry (e.g., [9.1]).

\section*{9.2 Hertzian Contact Stresses}

The first successful analysis of contact stresses is attributed to Hertz [9.2]. This analysis gave the dimensions of the contact area and the pressure distribution over that area. These quantities permit the computation of the displacements and stresses in the neighborhood of the region of contact. Belajev [9.3] and Thomas and Hoersch [9.4] performed important calculations of the stress fields in contacting solids. Discussions of the analysis of contact stresses can be found in the literature (e.g., [9.5, 9.6]). Tabulations of formulas applicable to special cases of contacting bodies can be found in such references as [9.7] and [9.8].

\subsection*{Two Bodies in Point Contact}

Figure 9-2 shows sections of two solid bodies with curved surfaces that are in contact. Before a force is applied to press the bodies together, they touch at one point only. When a force \(F\) is applied, elastic compression occurs near the initial point of contact, and a flat area of contact is formed. This area is tangent to the undeformed surfaces of the two solids and is perpendicular to the line of action of the force \(F\). The curvature of a surface is characterized at any point by the maximum and minimum values of the radii of curvature \(R'\) and \(R\). The two planes are orthogonal and contain \(R'\) and \(R\) and the surface normal. A radius of curvature of the surface of a body is
Figure 9-2: Two elastic solids in contact: (a) contact configuration; (b) before loading; (c) after loading, $xy$ axes coincide with major and minor axes of elliptical contact area (hatched area); (d) displacement of contacting points $M_1$ and $M_2$ and rigid distance of approach $d = d_1 + d_2$.

taken to be positive at a point if the corresponding center of curvature lies within the solid body; otherwise, the radius is negative. Quantities with the subscript 1 refer to the top body of Fig. 9-2a and those with the subscript 2 refer to the bottom solid. The two solids are assumed to be elastic, isotropic, and homogeneous; also, the contacting surfaces are smooth and free of frictional or adhesive forces. The four principal radii of curvature of the two surfaces at the point of contact are large compared to the dimensions of the contact area, and plastic deformation is ignored.
The coordinate system \((x, y, z)\) is aligned such that the \(xy\) plane lies tangent to the undeformed surfaces at the initial point of contact and such that the \(z\) axis coincides with the line of action of the force \(F\). Before deformation, suppose that the surfaces of the two bodies are approximately quadratic near the point of contact:

\[
\begin{align*}
  z_1 &= A_1 x^2 + B_1 y^2 + C_1 xy \\
  z_2 &= A_2 x^2 + B_2 y^2 + C_2 xy
\end{align*}
\]

(9.1)  
(9.2)

where \(z_1\) and \(z_2\) are the perpendicular distances from the tangent plane to any point on the surfaces of body 1 and body 2 near the point of contact, respectively, in the \(z\) direction (Fig. 9-2b). After deformation, two points that come into contact will have moved a distance

\[
  z_1 + z_2 = (A_1 + A_2) x^2 + (B_1 + B_2) y^2 + (C_1 + C_2) xy
\]

(9.3a)

Under the assumption that each pair of contacting points was initially on opposite ends of a line parallel to the \(z\) axis, all points with the same value of \(z_1 + z_2\) lie on an ellipse, and the perimeter of the contact area is elliptical. To eliminate the cross term in Eq. (9.3a), the \(x, y\) coordinates may be rotated to coincide with the major and minor axes of the elliptical contact area (Fig. 9-2c). Thus, Eq. (9.3a) can be rewritten as

\[
  z_1 + z_2 = Ax^2 + By^2
\]

(9.3b)

where \(A = A_1 + A_2\) and \(B = B_1 + B_2\).

Far from the contact area, material points of the two bodies are unaffected by elastic compressive deformation. These two regions will approach each other by a constant distance \(d\). This distance is a net rigid-body displacement of the two regions. Let \(w_1\) and \(w_2\) denote the local elastic displacements of points on the contacting surfaces. Take \(w_1\) and \(w_2\) as positive for compressive displacements (i.e., for displacements into the original configuration of the solid on the surface of which the point lies). The displacement of contacting points is given by

\[
  d - (w_1 + w_2) = z_1 + z_2 = Ax^2 + By^2
\]

(9.4)

where \(d = d_1 + d_2\) (Fig. 9-2d). This \(d\) is referred to as the rigid approach of two bodies. From geometric considerations [9.9] the constants \(A\) and \(B\) are functions of the four principal radii of curvature of the two undeformed surfaces and of the orientation of the principal planes of curvature of body 1 with respect to those of body 2 (Fig. 9-2c):

\[
  A = \frac{1}{4} \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R'_1} + \frac{1}{R'_2} \right) - \frac{1}{4} \left[ \left( \frac{1}{R_1} - \frac{1}{R'_1} \right) + \left( \frac{1}{R_2} - \frac{1}{R'_2} \right) \right]^2 \\
  - \left[ 4 \left( \frac{1}{R_1} - \frac{1}{R'_1} \right) \left( \frac{1}{R_2} - \frac{1}{R'_2} \right) \sin^2 \theta \right]^{1/2}
\]

(9.5)
\[ B = \frac{1}{4} \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R'_1} + \frac{1}{R'_2} \right) + \frac{1}{4} \left\{ \left[ \left( \frac{1}{R_1} - \frac{1}{R'_1} \right) + \left( \frac{1}{R_2} - \frac{1}{R'_2} \right) \right] \right\}^2 \\
- \left[ 4 \left( \frac{1}{R_1} - \frac{1}{R'_1} \right) \left( \frac{1}{R_2} - \frac{1}{R'_2} \right) \sin^2 \theta \right]^{1/2} \tag{9.6} \]

where \( \theta \) is the angle between the planes of maximum (or minimum) curvature of the two contacting bodies (Fig. 9-2c). The displacements \( w_1 \) and \( w_2 \) are found by superposition using Boussinesq’s solution [9.8] for a semi-infinite body subjected to a concentrated normal force at the boundary surface (the \( x, y \) plane). This approach neglects the curvature of the surfaces outside of the contact area:

\[ w_1 + w_2 = \left( \frac{1 - v_1^2}{\pi E_1} + \frac{1 - v_2^2}{\pi E_2} \right) \int \int_{A_c} \frac{p \, dA_c}{r} = d - Ax^2 - By^2 \tag{9.7} \]

In Eq. (9.7), \( p \, dA_c \) is considered to be a point force acting at a point \((x', y')\) in the contact area. The variables \( w_1 \) and \( w_2 \) are elastic compressive deformations at a point \((x, y)\) in the contact area. The variable \( r \) is the distance between \((x', y')\) and \((x, y)\). Boussinesq’s solution for the displacement \( dw_1 \) at \((x, y)\) due to a point force \( p \, dA_c \) at \((x', y')\) is

\[ dw_1 = \frac{1 - v_1^2}{\pi E_1} \frac{p \, dA_c}{r} = \frac{1 - v_1^2}{\pi E_1} \frac{p(x', y') \, dx' \, dy'}{\sqrt{(x - x')^2 + (y - y')^2}} \]

Of course, \( dw_2 \) is given by a similar equation.

To find the total displacement caused by the pressure \( p \) over the contact area, the elemental displacements are superimposed by integrating over the contact area \( A_c \) as shown in Eq. (9.7). Hertz found that Eq. (9.7) is satisfied if \( p(x, y) \) is given by

\[ p = p_0 \sqrt{1 - \left( \frac{x^2}{a^2} \right) - \left( \frac{y^2}{b^2} \right)} \tag{9.8} \]

in which \( a \) is the semiminor axis and \( b \) the semimajor axis of the contact ellipse (Fig. 9-2c). The distribution of pressure is semiellipsoidal with a maximum pressure \( p_0 \) at the center of the contact area:

\[ p_0 = \frac{3F}{2\pi ab} \tag{9.9} \]

It is apparent that the maximum pressure is 1.5 times the average pressure \( \left[ \frac{F}{(\pi ab)} \right] \). In general, the determination of the axes of the contact ellipse and of the distance of approach involves the evaluation of elliptic integrals [9.9].

Reference [9.9] contains compiled graphs for computing the quantities of interest in a contact problem. Figures 9-3 and 9-4 plot coefficients used in determining these quantities for values of \( B/A \) from 1 to 10, 000. The quantity \( C_b \) is used to compute \( b \) from the equation.
Figure 9-3: Coefficients for bodies in contact. From [9.9], with permission.

\[ b = C_b (F \Delta)^{1/3} \]  \hspace{1cm} (9.10a)

where

\[ \Delta = \left( \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right) \frac{1}{A + B} = \gamma \frac{1}{A + B} \]  \hspace{1cm} (9.10b)

where

\[ \gamma = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \]

Define the quantity \( k \) to be the ratio of the minor to major axes of the contact ellipse

\[ k = b/a \]  \hspace{1cm} (9.10c)

Once \( k \) and \( b \) are known \( a = b/k \) can be obtained. The displacement \( d \) is found by using the quantity \( C_d \):

\[ d = C_d (F/\pi)(A + B)/(b/\Delta) \]  \hspace{1cm} (9.11)
Figure 9-4: Coefficients for bodies in contact. From [9.9], with permission.

From knowledge of the dimensions of the contact area and the pressure distribution over it, Thomas and Hoersch [9.4] derived expressions for the principal stresses along the $z$ axis within the contacting solids. These formulas involve the evaluation of elliptic integrals. For any value of $B/A$, Fig. 9-3 or 9-4 can be used to compute the maximum compressive stress $(\sigma_c)_{\text{max}}$ that occurs at the origin, the maximum shear stress $\tau_{\text{max}}$ that occurs within the bodies, the maximum octahedral shear stress $(\tau_{\text{oct}})_{\text{max}}$, and the distance $Z_s$ from the contact area at which the maximum shear
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stresses occur. The curves are strictly accurate when \( \nu = 0.25 \), but the dependence of these quantities on \( \nu \) is weak:

\[
\sigma_c = (\sigma_c)_{\text{max}} = -C_\sigma (b/\Delta) \tag{9.12}
\]

\[
\tau_{\text{max}} = C_\tau (b/\Delta) \tag{9.13}
\]

\[
\tau_{\text{oct}} = (\tau_{\text{oct}})_{\text{max}} = C_{\text{oct}} (b/\Delta) \tag{9.14}
\]

\[
Z_s = C_z b \tag{9.15}
\]

The formulas above are summarized in Table 9-1. Sometimes the values of coefficients \( C_\sigma, C_\tau, C_b \), and so on. are difficult to read from Figs. 9-3 and 9-4. This problem can be avoided by using Table 9-2 for contact stress analyses. The coefficients \( n_a, n_b, n_c, n_d \), which appear in Table 9-2, can be taken from Table 9-3. The formulas used to calculate Table 9-3 are given in Table 9-4.

In many cases of practical interest, surface roughness, local yielding, friction, lubrication, thermal effects, and residual stresses will result in conditions that invalidate the Hertzian analysis. Consequently, the stresses computed according to Hertz’s analysis must often be regarded as guidelines that are correlated with experimental failure tests to find allowable stress limits.

The following section contains several examples of the computation of Hertzian contact stresses. Formulas pertinent to a number of special cases are listed in Table 9-2. In addition, Table 9-2 also provides some solutions of problems for contact stresses when the surfaces are not curved.

**Example 9.1 Wheel on a Rail** A steel wheel of radius 45 cm rests on a steel rail that has a radius of curvature of 35 cm (Fig. 9-5). The wheel supports a load of 40 000 N. To find the dimensions of the contact area, the maximum stresses in the

![Figure 9-5: Wheel on rail for Example 9.1 (crossed cylinders).](image)
contact region, and the distance below the contact surface at which the maximum shear stress and octahedral shear stress occur, the constants $A$ and $B$ must first be evaluated. Denoting the wheel as body 1 and the railhead as body 2, the principal radii of curvature are $R_1 = 45$ cm, $R_1' = \infty$, $R_2 = 35$ cm, and $R_2' = \infty$. The angle between the principal planes of the two bodies is $90^\circ$. The physical constants of steel are $E = 200$ GPa, $\nu = 0.29$.

From Eqs. (9.5) and (9.6),

$$A = \frac{1}{4} \left( \frac{1}{0.45} + \frac{1}{0.35} \right) - \frac{1}{4} \left[ \left( \frac{1}{0.45} + \frac{1}{0.35} \right)^2 - 4 \left( \frac{1}{0.45} \right) \left( \frac{1}{0.35} \right) \right]^{1/2}$$

$$= 1.2698 - 0.1587 = 1.111 \text{ m}^{-1}$$

(1)

$$B = 1.2698 + 0.1587 = 1.428 \text{ m}^{-1}$$

(2)

$$B/A = 1.428/1.111 = 1.285$$

(3)

When both bodies have the same physical properties, Eq. (9.10b) becomes

$$\Delta = \frac{2(1 - \nu^2)}{E(A + B)} = \frac{2 \left[1 - (0.29)^2\right]}{(2.0 \times 10^{11})(1.111 + 1.428)} = 3.607 \times 10^{-12} \text{ m}^3/\text{N}$$

(4)

From knowledge of $B/A$, the constants for use in determining stresses and lengths are read from Fig. 9-3, $C_b = 0.84$, $k = 0.85$, $C_\sigma = 0.69$, $C_\tau = 0.22$, $C_{oct} = 0.21$, $C_z = 0.5$, $C_d = 2.2$. The semiminor axis of the contact ellipse is given by [Eq. (9.10a)]

$$b = C_b(F\Delta)^{1/3} = 0.84 \left[ (40.000)(3.607 \times 10^{-12}) \right]^{1/3}$$

$$= 0.00441 \text{ m} = 4.41 \text{ mm}$$

(5)

The semimajor axis of the contact ellipse is [Eq. (9.10c)]

$$a = b/k = 0.00441/0.85 = 0.00519 \text{ m} = 5.19 \text{ mm}$$

(6)

The compressive stress at the center of the contact ellipse (i.e., the maximum principal stress) becomes [Eq. (9.12)]

$$\sigma_c = -C_\sigma (b/\Delta) = -0.69(0.00441/3.607 \times 10^{-12}) = -843.6 \text{ MPa}$$

(7)

The maximum shear stress is [Eq. (9.13)]

$$\tau_{\text{max}} = C_\tau (b/\Delta) = 269.0 \text{ MPa}$$

(8)

The maximum octahedral shear stress is given by [Eq. (9.14)]

$$\tau_{\text{oct}} = C_{\text{oct}} (b/\Delta) = 256.8 \text{ MPa}$$

(9)
The distance below the center of the contact area at which the two maximum shear stresses occur is found to be [Eq. (9.15)]

\[ Z_s = C_z b = 0.5(0.00441) = 0.002205 \text{ m} = 2.205 \text{ mm} \] (10)

Finally, the rigid approach of the two bodies becomes [Eq. (9.11)]

\[
d = C_d \left( \frac{F A + B}{\pi b/\Delta} \right) = (2.2)(4.0 \times 10^4) \left( \frac{1.111 + 1.428}{\pi} \frac{1}{0.00441}/(3.607 \times 10^{-12}) \right)
\]

\[ = 0.0582 \text{ mm} \] (11)

This problem also can be solved by using the formulas of Table 9-2. This is a contact stress problem of cylinders crossed at right angles. The formulas in case 2d apply. Then

\[ \gamma = 2 \frac{1 - v^2}{E} = 2 \frac{1 - 0.29^2}{2 \times 10^{11}} = 9.159 \times 10^{-12} \text{ m}^2/\text{N} \]

\[ K = \frac{D_1 D_2}{D_1 + D_2} = \frac{0.90 \times 0.70}{0.90 + 0.70} = 0.3938 \text{ m} \]

\[ B = 1/D_2 = 1/0.70 = 1.429 \text{ m}^{-1} \] (12)

\[ A = 1/D_1 = 1/0.90 = 1.111 \text{ m}^{-1} \]

\[ A/B = 0.70/0.90 = 0.7778 \]

From Table 9-3,

\[ n_a = 1.089, \quad n_b = 0.9212, \quad n_c = 0.9964, \quad n_d = 0.9964 \] (13)

The semimajor axis of the contact ellipse is

\[ a = 0.909 n_a (FK\gamma)^{1/3} \]

\[ = 0.909(1.089) \left( 40000(0.3938)9.159 \times 10^{-12} \right)^{1/3} \]

\[ = 5.192 \times 10^{-3} \text{ m} = 5.192 \text{ mm} \] (14)

while the semiminor axis is

\[ b = 0.909 n_a (FK\gamma)^{1/3} \]

\[ = 0.909(0.9212) \left( 40000(0.3938)9.159 \times 10^{-12} \right)^{1/3} \]

\[ = 4.39 \times 10^{-3} \text{ m} = 4.39 \text{ mm} \] (15)
The maximum compressive stress becomes

\[
\sigma_c = 0.579n_c \left[ \frac{F}{(K^2\gamma^2)} \right]^{1/3}
\]

\[
= 0.579 \times 0.9964 \left( \frac{40000}{0.3938^2 \times 9.1592 \times 10^{-24}} \right)^{1/3} = 838.8 \text{ MPa} \tag{16}
\]

The rigid approach of the two bodies is given as

\[
d = 0.825n_d \left( \frac{F^2\gamma^2}{K} \right)^{1/3}
\]

\[
= 0.825(0.9964) \left( \frac{40000^2 \times 9.1592 \times 10^{-24}}{0.3938} \right)^{1/3}
\]

\[
= 0.05742 \text{ mm} \tag{17}
\]

**Example 9.2 Ball Bearing**  At the contact region of the ball bearing system shown in Fig. 9-6, find the maximum principal stress, the maximum shearing stress, the maximum octahedral shearing stress, the dimensions of the area of contact, and the distance from the point of contact to the point along the force direction where the stresses occur. Assume that \( E = 200 \text{ GN/m}^2 \) and \( \nu = 0.3 \).
The radii of concern are given in Fig. 9-6 as

\[ R_1 = \frac{1}{2}d_0 = 19 \text{ mm}, \quad R'_1 = \frac{1}{2}d_0 = 19 \text{ mm} \]
\[ R_2 = -r = -20 \text{ mm}, \quad R'_2 = \frac{1}{2}D = 100 \text{ mm} \]

From Eqs. (9.5) and (9.6),

\[
A = \frac{1}{4} \left( \frac{1}{0.019} - \frac{1}{0.020} + \frac{1}{0.019} + \frac{1}{0.100} \right)
\frac{1}{4} \left[ \left( \frac{1}{0.019} - \frac{1}{0.019} \right) + \left( -\frac{1}{0.020} - \frac{1}{0.100} \right) \right]^2
- 4 \left( \frac{1}{0.019} - \frac{1}{0.019} \right) \left( -\frac{1}{0.020} - \frac{1}{0.100} \right) \sin^2(0) \right]^{1/2} = 1.316 \text{ m}^{-1}
\]

\[ B = 31.32 \text{ m}^{-1} \]

Then

\[
\frac{B}{A} = 23.78
\]
\[
\Delta = \frac{2}{A + B} \left( 1 - \nu^2 \right) = \frac{2(1 - 0.3^2)}{(31.32 + 1.316)(200 \times 10^9)}
= 2.79 \times 10^{-13} \text{ m}^3/\text{N}
\]

From Fig. 9-3, the coefficients are found. The variables of interest are then computed using the appropriate formulas:

\[ k = 0.13, \quad C_{\text{oct}} = 0.27, \quad C_{\tau} = 0.3 \]
\[ C_b = 0.394, \quad C_\sigma = 1.0, \quad C_z = 0.8 \]
\[ b = C_b (F \Delta)^{1/3} = 0.394[(4500)(2.79 \times 10^{-13})]^{1/3} = 4.250 \times 10^{-4} \text{ m} \]
\[ = 0.425 \text{ mm} \]
\[ a = 4.250 \times 10^{-4}/0.13 = 3.269 \times 10^{-3} \text{ m} = 3.269 \text{ mm} \]
\[ b/\Delta = 4.250 \times 10^{-4}/(2.79 \times 10^{-13}) = 1523 \text{ MPa} \]
\[ \sigma_c = -C_\sigma (b/\Delta) = (-1.0)(1523) = -1523 \text{ MPa} \]
\[ \tau_{\text{max}} = C_{\tau} (b/\Delta) = 0.3(1523) = 456.9 \text{ MPa} \]
\[ \tau_{\text{oct}} = C_{\text{oct}} (b/\Delta) = (0.27)(1523) = 411.2 \text{ MPa} \]
\[ Z_s = C_z b = (0.8)(4.250 \times 10^{-4}) = 3.40 \times 10^{-4} \text{ m} = 0.34 \text{ mm} \]
Alternatively, use the formulas of case 1e of Table 9-2:

\[
\gamma = 2 \frac{1 - \nu^2}{E} = 2 \frac{1 - 0.3^2}{2 \times 10^{11}} = 0.91 \times 10^{-11} \text{ m}^2/\text{N}
\]

\[
K = \frac{1}{2/R_1 - 1/R_2 + 1/R_3} = \frac{1}{2/0.019 - 1/0.02 - 1/0.10} = 0.01532
\]

\[
A = \frac{1}{2} \left( \frac{1}{R_1} - \frac{1}{R_2} \right) = \frac{1}{2} \left( \frac{1}{0.019} - \frac{1}{0.020} \right) = 1.316 \text{ m}^{-1}
\]

\[
B = \frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_3} \right) = \frac{1}{2} \left( \frac{1}{0.019} + \frac{1}{0.1} \right) = 31.32 \text{ m}^{-1}
\]

\[
A/B = 0.04202
\]

From Table 9-3,

\[
na = 3.385, \quad nb = 0.4390, \quad nc = 0.6729, \quad nd = 0.6469 \quad (7)
\]

The semimajor axis of the contact ellipse is given by

\[
a = 1.145na(FK\gamma)^{1/3} = 1.145 \times 3.385 \times \left[ 4500(0.91 \times 10^{-11})0.01532 \right]^{1/3}
\]

\[
= 3.31 \times 10^{-3} \text{ m} = 3.31 \text{ mm} \quad (8)
\]

and the semiminor axis is

\[
b = 1.145nb(FK\gamma)^{1/3} = 0.4303 \text{ mm} \quad (9)
\]

Furthermore, the maximum compressive stress is

\[
\sigma_c = 0.365nc \left[ F/(K^2\gamma^2) \right]^{1/3} = 1508 \text{ MPa} \quad (10)
\]

and the rigid approach of the two bodies becomes

\[
d = 0.655nd(F^2\gamma^2/K)^{1/3} = 0.02027 \text{ mm} \quad (11)
\]

Example 9.3 Wheel–Rail Analyses Consider again the wheel and rail shown in Fig. 9-5. In Example 9.1, the maximum octahedral shear stress was found to be 256.8 MPa and to be located 0.22 cm below the initial contact point. Suppose now that the rail steel has a tensile yield strength of 413.8 MPa.

1. Determine whether yielding occurs in the rail according to the maximum octahedral shear stress yield theory (equivalent to the von Mises–Hencky theory).
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The octahedral shear stress at yield is [Eq. (3.24b)]

\[
(\tau_{oct})_{ys} = \frac{1}{3}(2\sigma_{ys}^2)^{1/2} = \frac{1}{4}\sqrt{2}\ \sigma_{ys} = \frac{1}{4}\sqrt{2}(413.8 \text{ MPa}) = 195.07 \text{ MPa} \tag{1}
\]

Since the maximum octahedral shear stress computed by elastic theory exceeds the yield value, yielding does occur in the rail.

2. Find to what value the load must be reduced so that the computed maximum octahedral shear stress equals the yield point value. From Eqs. (9.14) and (9.10a),

\[
\tau_{oct} = C_{oct}(b/\Delta), \quad b = C_b(F/\Delta)^{1/3}
\]

Hence

\[
\tau_{oct} = C_{oct}C_b(F/\Delta)^{1/3}/\Delta, \quad (\tau_{oct})^3 = (C_{oct}C_b/\Delta)^3 \Delta F = F(C_{oct}C_b)^{3/2}/\Delta^2
\]

\[
F = \left[\Delta^2/(C_{oct}C_b)^3\right](\tau_{oct})^3 \tag{2}
\]

Since \(C_{oct}, C_b, \) and \(\Delta\) do not depend on \(F\), the yield load \(F_{ys}\) is calculated as

\[
F_{ys} = \frac{(3.607 \times 10^{-12})^2(195.07 \times 10^6)^3}{(0.2)^3(0.84)^3} = 20\ 367 \text{ N} \tag{3}
\]

Therefore, to reduce the maximum octahedral shear stress to the yield value, the applied load of 40 000 N must virtually be halved.

3. Suppose that the wheel–rail combination must be operated with a safety factor of 2 (i.e., the maximum octahedral shear stress must be one-half the value that causes yield). Compute the maximum value the load may take under this restriction.

Since maximum octahedral shear stress varies directly as the cube root of the applied load, to halve the stress, the load must decrease by a factor of \((\frac{1}{2})^3\), or \(\frac{1}{8}\). Since a load of 20 367 N corresponds to a maximum octahedral shear stress exactly at the yield point, the force

\[
F_2 = \frac{1}{8}20\ 367 \text{ N} = 2545.9 \text{ N} \tag{4}
\]

would result in the maximum octahedral shear stress being one-half the yield value.

4. Suppose that the operating load must be 20 367 N. Find by what common factor the radii \(R_1\) and \(R_2\) must be increased in order that the maximum octahedral shear stress be one-half the value that causes yielding.

The stress \(\tau_{oct}\) in terms of the load \(F\) is given by (2). With \(A\) and \(B\) defined by Eqs. (9.5) and (9.6), changing \(R_1\) and \(R_2\) by the same factor does not affect \(B/A\), so \(C_{oct}\) and \(C_b\) of Figs. 9-3 and 9-4 remain constant. Similarly, \(\gamma\) depends only on \(E\) and \(\nu\) so that \(\Delta\) of Eq. (9.10b) changes only as a result of \(A + B\). Let \(\tau, A, B, R_1, R_2\) be the values of variables under conditions described in question 2 and \(\tau_{oct}^*, A^*, B^*, R_1^*, R_2^*\) be the conditions with \(R_1\) and \(R_2\) altered by a factor, say \(\lambda\). We require that

\[
\tau_{oct}^* = \frac{1}{2}\tau_{oct}, \quad R_1^* = \lambda R_1, \quad R_2^* = \lambda R_2
\]

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From Eqs. (9.5) and (9.6),

\[
A + B = \frac{1}{4} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + \frac{1}{4} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) = \frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)
\]

\[
A^* + B^* = \frac{1}{2} \left( \frac{1}{R_1^*} + \frac{1}{R_2^*} \right) = \frac{1}{2\lambda} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)
\]

and from (2),

\[
\tau_{oct}^3 = F(c_G c_b)^3 / \Delta^2, \quad \tau_{oct}^* = F(c_G c_b)^3 / \Delta^*^2
\]

Then

\[
\frac{\tau_{oct}^3}{\tau_{oct}^*} = \frac{\Delta^*^2}{\Delta^2} = \frac{[\gamma/(A^* + B^*)]^2}{[\gamma/(A + B)]^2} = \frac{(A + B)^2}{(A^* + B^*)^2} = \frac{1/4 (1/R_1 + 1/R_2)^2}{(1/4\lambda^2) (1/R_1 + 1/R_2)^2}
\]

Thus,

\[
\frac{\tau_{oct}}{\tau_{oct}^*} = \lambda^2 \quad \text{or} \quad \tau_{oct}^* = \tau_{oct} / \lambda^{2/3}
\]

Since

\[
\tau_{oct}^* = \frac{1}{2} \tau_{oct}
\]

it follows that

\[
2^3 \tau_{oct}^3 / \tau_{oct}^3 = \lambda^2 \quad \text{or} \quad \lambda = \sqrt{8}
\]

To check, we find

\[
R_1^* = \sqrt{8}(45) = 127.28 \text{ cm}, \quad R_2^* = \sqrt{8}(35) = 98.995 \text{ cm}
\]

\[
(A + B)^* = \frac{1}{2} \left( \frac{1}{1.2728} + \frac{1}{0.98995} \right) = 0.8979 \text{ m}^{-1}
\]

\[
\Delta^* = \frac{2 [1 - (0.29)^2]}{(2 \times 10^{11})(0.8979)} = 1.020 \times 10^{-11} \text{ m}^3 / \text{N}
\]

\[
\tau_{oct}^* = \frac{(20367)(0.2)^3(0.84)^3}{(1.02 \times 10^{-11})^2} = 9.282 \times 10^{23} (\text{N/m}^2)^3
\]

\[
\tau_{oct}^* = 9.755 \times 10^7 \text{ Pa} \quad \text{or} \quad 97.55 \text{ MPa}
\]

Since the yield value of maximum octahedral shear is 195.07 MPa and 97.55 is one-half of the yield value, increasing \( R_1 \) and \( R_2 \) by a factor of \( \sqrt{8} \) decreases the maximum octahedral shear stress by one-half.
5. Suppose that the operating load is fixed at 20,367 N and that the rail and wheel radii are fixed at 35 and 45 cm, respectively. Find by what factor the tensile strength of the steel must be increased to make the maximum octahedral shear stress one-half the yield point value.

Since $E$ and $\nu$ of steel are essentially constant for steels of all strengths and $A + B$ is determined by the fixed radii of rail and wheel, the quantity $\Delta$ in Eq. (9.10b) is a fixed value. Therefore, from (5), the maximum octahedral shear stress would remain at 195.07 MPa for all steels. Because tensile yield strength and octahedral shear stress at yield are directly proportional, doubling the tensile strength would result in the maximum octahedral shear stress being one-half the value that causes yield. From (1), the strength of the steel would be increased to

$$\sigma_{ys} = \frac{3}{\sqrt{2}}(\tau_{oct})_{ys} = \frac{3}{\sqrt{2}}(2 \times 195.07) = 827.6 \text{ MPa} \quad (8)$$

6. Determine for which of the three quantities (load, radii of curvature, or steel strength) would a change be most effective in producing a system with an acceptable value of maximum octahedral shear stress.

Reducing the load is most ineffective in reducing the maximum octahedral shear stress because, from (2), the stress varies directly as the cube root of the load. When the radii of curvature are increased in constant proportion, the maximum octahedral shear stress varies inversely as the two-thirds power of the radii factor $\lambda$ [see (6)]; hence changing the radii is more effective than changing the load. However, if large reductions in stress are required, it is doubtful that the necessarily large changes in radii ($\lambda = \sqrt{8} = 2.83$-fold increase for a halving of the shear stress) would be feasible. It appears from the previous question that increasing the tensile strength of the material of construction is the most effective alternative when the stress is significantly higher than an acceptable level.

### Two Bodies in Line Contact

Two bodies in contact along a straight line before loading are said to be in line contact. For instance, a line contact occurs when a circular cylinder rests on a plane or when a small circular cylinder rests inside a larger hollow cylinder. In these line contact cases, Eqs. (9.5) and (9.6) become

$$A = 0, \quad B = \frac{1}{2}(1/R_1 + 1/R_2)$$

and

$$B/A = \infty \quad (9.16)$$

It can be shown that in this case, the quantity $k$ in Eq. (9.10c) approaches zero. When a distributed load $q$ (force/length) is applied, the area of contact is a long narrow rectangle of width $2b$ in the $x$ direction and a length $2a$ in the $y$ direction.
The maximum principal stresses occurring at the surface of contact are \( \sigma_x = -b/\Delta, \quad \sigma_y = -2\nu(b/\Delta), \quad \sigma_z = -b/\Delta \) \hspace{1cm} (9.17a)

Thus,

\[ \sigma_{\text{max}} = -b/\Delta \] \hspace{1cm} (9.17b)

where

\[ b = \sqrt{2q\Delta/\pi} \] \hspace{1cm} (9.18a)

\[ \Delta = \frac{1}{1/(2R_1) + 1/(2R_2)} \left( \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right) \] \hspace{1cm} (9.18b)

The maximum shear stress is

\[ \tau_{\text{max}} = 0.300(b/\Delta) \] \hspace{1cm} (9.19)

at the depth \( Z_s/b = 0.7861 \).

The maximum octahedral shear stress occurs at the same point as the maximum shear. The value is

\[ \tau_{\text{oct}} = 0.27(b/\Delta) \] \hspace{1cm} (9.20)

For the case of line contact, Eqs. (9.12)–(9.15) still apply. The coefficients \( C_\sigma, C_\tau, C_{\text{oct}}, C_z \) can also be found from Figs. 9-3 and 9-4 by selecting values of \( B/A \) greater than 50.

**Contact Stress with Friction**

For the case of two cylinders with longitudinal axes parallel, Smith and Liu \[9.11\] examined the modification of the contact stress field caused by the presence of surface friction. Mindlin \[9.12\] showed that the tangential stresses have the same distribution over the contact areas as have the normal stresses. For impending sliding motion, the tangential stresses are linearly related to the normal stresses by a coefficient of friction. The total stress field is the resultant of the field due to normal surface stresses plus the field due to tangential surface stresses. The degree to which tangential surface stresses change the distribution caused by normal surface stresses depends on the magnitude of the coefficient of friction. The changes in the maximum contact stresses with the coefficient of friction are provided in Table 9-5.

The presence of friction may lead to changes from a compressive stress to a stress that varies from tensile to compressive over the area of contact. The creation of tensile stresses in the contact zone is thought to contribute to fatigue failure of bodies subject to cyclic contact stresses. Smith and Liu found in addition that if the coeffi-
9.2 HERTZIAN CONTACT STRESSES

Cient of friction was 0.1 or greater, the point of maximum shear stress occurs on the contact surface rather than below it.

**Example 9.4 Contact Stress in Cylinders with Friction**  Consider two steel cylinders each 80 mm in diameter and 150 mm long mounted on parallel shafts and loaded by a force $F = 80$ kN (Fig. 9-7). The two cylinders ($E = 200$ GPa and $v = 0.29$) are rotated at slightly different speeds so that the cylinder surfaces slide across each other. If the coefficient of sliding friction is $\mu = \frac{1}{3}$, determine the maximum compressive principal stress $\sigma_c$, the maximum shear stress $\tau_{\text{max}}$, and the maximum octahedral shear stress $\tau_{\text{oct}}$.

The value of the required quantities are obtained from Table 9-5 for $\mu = \frac{1}{3}$,

$$\begin{align*}
(\sigma_c)_{\text{max}} &= -1.40(b/\Delta) \\
\tau_{\text{max}} &= 0.435(b/\Delta) \\
\tau_{\text{oct}} &= 0.368(b/\Delta)
\end{align*}$$

(1) (2) (3)

where $b$ and $\Delta$ are given by Eqs. (9.18a) and (9.18b) with $q = F/\ell$:

$$\Delta = 2R \frac{1 - v^2}{E} = \frac{2(0.040)(1 - 0.29^2)}{200 \times 10^9} = 3.664 \times 10^{-13} \text{ m}^3/\text{N}$$

(4)

$$b = \left( \frac{2F\Delta}{\ell\pi} \right)^{1/2} = \sqrt{\frac{2(80 \times 10^3)(3.664 \times 10^{-13})}{0.150\pi}} = 0.3527 \times 10^{-3} \text{ m}$$

(5)

$$b/\Delta = 962.6 \text{ MPa}$$

(6)

![Figure 9-7: Example 9.4.](image-url)
Substitution of these values into (1), (2), and (3) leads to

\[
(\sigma_c)_{\text{max}} = -1.40 \times 962.6 = -1347.6 \text{ MPa}
\]
\[
\tau_{\text{max}} = 0.435 \times 962.6 = 418.7 \text{ MPa},
\]
\[
\tau_{\text{oct}} = 0.368 \times 962.6 = 354.2 \text{ MPa}
\]

It can be seen from Table 9-5 that the friction force with \( \mu = \frac{1}{3} \) increases the maximum compressive principal stress by 40%, the maximum shear stress by 45%, and the maximum octahedral shear stress by 35% relative to the case with \( \mu = 0 \).

### 9.3 CONTACT FATIGUE

A machine part subjected to contact stresses usually fails after a large number of load applications. The failure mode is that of crack initiation followed by propagation until the part fractures or until pits are formed by material flaking away. Buckingham measured the surface fatigue strengths of materials subjected to contact loads [9.13]. His results showed that hardened steel rollers did not have a fatigue limit for contact loading. Cast materials, however, did show a fatigue limit for contact loads.

### 9.4 ROLLING CONTACT

When two bodies roll over each other, the area of contact will in general be divided into a region of slip and a region of adhesion. In the region of slip the tangential force is related to the normal force by a coefficient of friction. Under conditions of free rolling no region of slip exists, and surface friction dissipates no energy. If gross sliding occurs, no region of adhesion exists.

When both regions are present, the motion is termed creep, creep ratio, or creepage. The creepage is resolved into three components: longitudinal, lateral, and spin. Spin creepage occurs when a relative angular velocity about an axis normal to the contact zone exists between the two contacting bodies. Longitudinal and lateral creepage occurs when a relative circumferential velocity without gross sliding exists between the contacting bodies. The forces and moments transmitted between two contacting bodies due to creepage are very important in wheel–rail contact problems. Vermeulen and Johnson [9.14] suggested a nonlinear law that does not account for spin creepage. Kalker has proposed a linear law relating creepage to the transmitted forces and moments as well as nonlinear creep laws [9.15].

### 9.5 NON-HERTZIAN CONTACT STRESS

The simplest non-Hertzian contact problem is the case in which all conditions for Hertzian contact are met except that the surfaces cannot be approximated as a
second-degree polynomial near the point of contact. Singh and Paul [9.16] have described a numerical procedure for solving this type of problem. In this method a suitable contact area is first proposed; then the corresponding applied load, pressure distribution, and rigid approach are found.

A general treatment of the interfacial responses of contact problems as nonlinear phenomena is formulated using finite element approximations in Laursen [9.21].

9.6 NANOTECHNOLOGY: SCANNING PROBE MICROSCOPY

Technology has been rapidly evolving to smaller and smaller scales. This has led to “nanotechnology,” so named because the scale of research and development is on the order of one-billionth of a meter \((10^{-9} \text{ m})\). The goals of nanotechnology include both scaling down current materials to the nanolevel and the construction of materials atom by atom. In the past, the progress of nanotechnology has been limited because tools for such small-scale research did not exist. However, with the advent of scanning probe instruments, attaining the goals of nanotechnology is becoming realizable.

One example of a scanning probe instrument is an atomic force microscope (AFM). Atomic force microscopy is useful technology for the study of surface force exertions. The AFM consists of a cantilevered beam with a probe, or tip, attached to the end. The tip is run across a surface and deflects as it interacts with the surface. With the help of a surface scan by a piezoactuator, the cantilever deflection may be measured leading to the surface topography.

To understand what an AFM accomplishes, consider the following example. When you move your finger across different surfaces, each one exerts a different force on your finger. As a result, you can differentiate between wood and steel or between silk and rubber. For example, when you run your finger over silk, you experience very little resistance and your finger slides easily across the surface. However, when you rub your finger across rubber, you experience a much larger resistance. Similarly, an AFM measures forces that the sampled surface exerts on the scanning tip, only on a much smaller scale than your finger.

In addition to atomic force microscopes, other scanning probe instruments include scanning tunneling microscopes and magnetic force microscopes. With scanning tunneling microscopy, electrical currents between the probe and surface can be measured. Magnetic force microscopy uses a magnetic tip to test the magnetic properties of the sampled surface.

A summary of some contact theories used in conjunction with the AFM is provided in Ref. [9.22]. The tip-sample interactions can be modeled by a variety of models that are appropriate for certain materials and environments.

**Hertz Model**

The Hertz contact model is not appropriate for some AFM experiments, since the model is designed for high loads or low surface forces. It is assumed that there are
no surface or adhesion forces. The AFM tip would be a smooth elastic sphere, and the contact surface is rigid and flat. In practice, for most cases the AFM tip is stiffer than the contact surface and the Hertz model is not suitable for calculating deformations if the tip is assumed to be rigid.

**Sneddon’s Model**

If the contact surface is softer than the tip, Sneddon’s model may be appropriate. In this case, the tip is rigid and the contact surface is elastic. Also, there are no surface or adhesion forces. It is possible, when no surface forces are present to combine the Hertz and Sneddon models to compute the deformation of the tip and the contact surface.

**Derjaguin-Muller-Toporov Theory (DMT)**

This model permits surface forces, yet restricts the tip-contact surface geometry to be Hertzian. This leads to finite stresses at the contact periphery, although non-Hertzian deformation there is neglected. As a result the contact area may be underestimated. The area of contact, based on including forces acting between two bodies outside the contact region, increases under an applied positive force and decreases for a negative force.

**Johnson-Kendall-Roberts Theory (JKR)**

This model is suitable for a highly adhesive tip-surface system with low stiffness and large tip radii. There is a nonzero contact area for a zero load. During unloading a neck can be formed between the tip and the contact surface. The predicted surface forces may be quite low. Shortcomings of this approach include the predictions of infinite stresses at the edge of the contact area. With this theory the attractive forces act over a very small range.

**Maugis-Dugdale Model**

This model is appropriate for hard or soft materials and for contact surfaces with high or low energies. Adhesion is treated as traction over an annular region around the contact area. This model effectively bridges the DMT and JKR models by introducing a parameter $\lambda$ that compares the relative magnitude of the elastic deformation at pull-off forces and the effective range of the surface force.

**REFERENCES**

9-1  Summary of General Formulas for Contact Stresses 438
9-2  Formulas for Contact Stresses, Dimensions, and Contact Areas, and Rigid-Body Approaches 439
9-3  Parameters for Use with Formulas of Table 9-2 446
9-4  Equations for the Parameters of Table 9-3 448
9-5  Contact Stresses between Two Long Cylindrical Bodies in Line Contact Sliding against Each Other 449
### Table 9-1 Summary of General Formulas for Contact Stresses

**Notation**

- $R_i, R'_i$: minimum and maximum radii of curvature of two contacting surfaces $i = 1, 2$
- $F$: applied force
- $\theta$: angle between planes containing principal radii of curvature

\[
A = \frac{1}{4} \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R'_1} + \frac{1}{R'_2} \right)
\]

\[
-\frac{1}{4} \left\{ \left[ \left( \frac{1}{R_1} - \frac{1}{R'_1} \right) + \left( \frac{1}{R_2} - \frac{1}{R'_2} \right) \right]^2 - 4 \left( \frac{1}{R_1} - \frac{1}{R'_1} \right) \left( \frac{1}{R_2} - \frac{1}{R'_2} \right) \sin^2 \theta \right\}^{1/2}
\]

\[
B = \frac{1}{4} \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R'_1} + \frac{1}{R'_2} \right)
\]

\[
+ \frac{1}{4} \left\{ \left[ \left( \frac{1}{R_1} - \frac{1}{R'_1} \right) + \left( \frac{1}{R_2} - \frac{1}{R'_2} \right) \right]^2 - 4 \left( \frac{1}{R_1} - \frac{1}{R'_1} \right) \left( \frac{1}{R_2} - \frac{1}{R'_2} \right) \sin^2 \theta \right\}^{1/2}
\]

Compute $B/A$ and obtain coefficients $C_b, k, C_\sigma, C_\tau, C_{\text{oct}}, C_z, C_d$ from plots of Figs. 9-3 or 9-4. Then

\[
\gamma = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}, \quad \Delta = \gamma \frac{A + B}{A + B}
\]

#### Formulas for Stresses and Deformation

**Semiminor axis:**

\[
b = C_b(F\Delta)^{1/3}
\]

**Semimajor axis:**

\[
a = b/k
\]

**Maximum compressive stress:**

\[
(\sigma_c)_{\text{max}} = -C_\sigma(b/\Delta)
\]

**Maximum shear stress:**

\[
\tau_{\text{max}} = C_\tau(b/\Delta)
\]

**Maximum octahedral shear stress:**

\[
(\tau_{\text{oct}})_{\text{max}} = C_{\text{oct}}(b/\Delta)
\]

**Distance from contact area to location of maximum shear stress:**

\[
Z_s = C_z b
\]

**Distance of approach of contacting bodies:**

\[
d = C_d \frac{F \Delta + B}{b/\Delta}
\]
TABLE 9-2 FORMULAS FOR CONTACT STRESSES, DIMENSIONS, AND CONTACT AREAS, AND RIGID-BODY APPROACHES\textsuperscript{a}

<table>
<thead>
<tr>
<th>Case</th>
<th>Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Spheres</strong></td>
<td></td>
</tr>
<tr>
<td>Ia. Sphere on sphere</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( a = b = 0.721(FK\gamma)^{1/3} )</td>
</tr>
<tr>
<td></td>
<td>( \sigma_c = 0.918[F/(K^2\gamma^2)]^{1/3} )</td>
</tr>
<tr>
<td></td>
<td>( d = 1.040(F^2\gamma^2/K)^{1/3} )</td>
</tr>
</tbody>
</table>

\textsuperscript{a} Notation

\[\gamma = (1 - \nu_i^2)/E_i \]

\( a, b \) = semimajor axis and semiminor axis of contact ellipse, respectively

\( d \) = rigid distance of approach of contacting bodies or surface deformation

\( E_i \) = modulus of elasticity of object \( i \), \( i = 1 \) or \( 2 \)

\( F \) = force

\( p \) = pressure

\( q \) = distributed line load

\( \sigma_c \) = maximum compressive stress of contact area, \( (\sigma_c)_{\text{max}} \)

\( \nu_i \) = Poisson’s ratio of object \( i \), \( i = 1 \) or \( 2 \)

\( \tau \) = shear stress
<table>
<thead>
<tr>
<th>Case</th>
<th>Formulas</th>
</tr>
</thead>
</table>
| 1b. Sphere on flat plate | $a = b = 0.721(FK\gamma)^{1/3}$  
|                 | $\sigma_c = 0.918[F/(K^2\gamma^2)]^{1/3}$  
|                 | $d = 1.040(F^2\gamma^2/K)^{1/3}$           |
| $K = D_1$       |                                              |
| 1c. Sphere in spherical socket | $a = b = 0.721(FK\gamma)^{1/3}$  
|                 | $\sigma_c = 0.918[F/(K^2\gamma^2)]^{1/3}$  
|                 | $d = 1.040(F^2\gamma^2/K)^{1/3}$           |
| $K = \frac{D_1D_2}{D_2 - D_1}$ |                                              |
### TABLE 9.2  Formulas for Contact Stresses

| 1d. Sphere on a cylinder | \( a = 0.9088 n_a (F K \gamma)^{1/3} \) |
| | \( b = 0.9088 n_b (F K \gamma)^{1/3} \) |
| | \( \sigma_c = 0.579 n_c [F / (K^2 \gamma^2)]^{1/3} \) |
| | \( d = 0.825 n_d (F^2 \gamma^2 / K)^{1/3} \) |
| | \( A = \frac{1}{D_1} \quad B = \frac{1}{D_1} + \frac{1}{D_2} \quad K = \frac{D_1 D_2}{2D_2 + D_1} \) |

| 1e. Sphere in circular race | \( a = 1.145 n_a (F K \gamma)^{1/3} \) |
| | \( b = 1.145 n_b (F K \gamma)^{1/3} \) |
| | \( \sigma_c = 0.365 n_c [F / (K^2 \gamma^2)]^{1/3} \) |
| | \( d = 0.655 n_d (F^2 \gamma^2 / K)^{1/3} \) |
| | \( K = \frac{1}{R_1 - \frac{1}{R_2} + \frac{1}{R_3}} \quad A = \frac{1}{2} \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \quad B = \frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_3} \right) \) |

<p>| 1f. Sphere in cylindrical race | ( a = 1.145 n_a (F K \gamma)^{1/3} ) |
| | ( b = 1.145 n_b (F K \gamma)^{1/3} ) |
| | ( \sigma_c = 0.365 n_c [F / (K^2 \gamma^2)]^{1/3} ) |
| | ( d = 0.655 n_d (F^2 \gamma^2 / K)^{1/3} ) |
| | ( K = \frac{R_1 R_2}{2R_2 - R_1} \quad A = \frac{1}{2} \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \quad B = \frac{1}{2R_1} ) |</p>
<table>
<thead>
<tr>
<th>Case</th>
<th>Formulas</th>
</tr>
</thead>
</table>
| **2a. Cylinder on cylinder (axes parallel)** | $b = 0.798(q K \gamma)^{1/2}$  $\sigma_c = 0.798[q/(K \gamma)]^{1/2}$  
$\ell = \frac{2q}{\pi} \left[ \frac{1}{E_1} \ln \frac{D_1}{b} + 0.407 \right] + \frac{1}{E_2} \left( \ln \frac{D_2}{b} + 0.407 \right)$  
$K = \frac{D_1 D_2}{(D_1 + D_2)}$ |
| **2b. Cylinder on flat plate** | $b = 0.798(q K \gamma)^{1/2}$  $\sigma_c = 0.798[q/(K \gamma)]^{1/2}$  
$K = D_1$ |
| **2c. Cylinder in cylindrical socket** | $b = 0.798(q K \gamma)^{1/2}$ when $E_1 = E_2$ and $\nu_1 = \nu_2 = 0.3$  
$\sigma_c = 0.798[q/(K \gamma)]^{1/2}$  
$K = \frac{D_1 D_2}{D_2 - D_1}$  
$d = 1.82 \frac{q}{E}(1 - \ln b)$ |
### TABLE 9-2 Formulas for Contact Stresses

#### 2d.
Cylinders crossed at right angles

<table>
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<td>( a = 0.909n_a(FK\gamma)^{1/3} )</td>
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<td>( b = 0.909n_b(FK\gamma)^{1/3} )</td>
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<tr>
<td>( \sigma_c = 0.579n_c\left[F/(K^2\gamma^2)\right]^{1/3} )</td>
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<tr>
<td>( d = 0.825n_d(F^2\gamma^2/K)^{1/3} )</td>
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<tr>
<td>( K = \frac{D_1D_2}{D_1+D_2} )</td>
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#### 3.
Barrel in a circular race

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<td>( b = 1.145n_b(FK\gamma)^{1/3} )</td>
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<tr>
<td>( \sigma_c = 0.365n_c\left[F/(K^2\gamma^2)\right]^{1/3} )</td>
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<td>( d = 0.655n_d(F^2\gamma^2/K)^{1/3} )</td>
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<tr>
<td>( K = \frac{1}{\pi_1+\pi_2+\pi_3-\pi_4} )</td>
<td>( A = \frac{1}{2}\left(\frac{1}{\pi_2} - \frac{1}{\pi_4}\right) )</td>
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#### 4a.
Rigid knife edge on surface of semi-infinite plate line load \( q \)

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<tr>
<td>( \sigma_c = \sigma_r = \frac{2q}{\pi r} \cos(\alpha + \theta) )</td>
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<tr>
<td>( \sigma_\theta = \tau_{r\theta} = 0 )</td>
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---

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### TABLE 9-2 (continued) FORMULAS FOR CONTACT STRESSES, DIMENSIONS, AND CONTACT AREAS, AND RIGID-BODY APPROACHES

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<th>Case</th>
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<td><strong>4b.</strong></td>
<td>Concentrated force on surface of semi-infinite body</td>
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<td>At an elemental area perpendicular to z axis of any point Q, the resultant stress is ( \frac{3F \cos^2 \theta}{2\pi r^2} )</td>
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<td></td>
<td><img src="image" alt="Diagram" /></td>
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<td><strong>4c.</strong></td>
<td>Uniform pressure ( p ) over length ( \ell ) on surface of semi-infinite body</td>
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<td>At surface point ( O_1 ) outside loaded area ( d = \frac{2p}{\pi E} \left[ (\ell + x_1) \ln \frac{c}{\ell + x_1} - x_1 \ln \frac{d}{x_1} \right] + p\ell\frac{1-v}{\pi E} )</td>
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<td></td>
<td>At surface point ( O_2 ) underneath loaded area ( d = \frac{2p}{\pi E} \left[ (\ell - x_2) \ln \frac{c}{\ell - x_2} + x_2 \ln \frac{d}{x_2} \right] + p\ell\frac{1-v}{\pi E} )</td>
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<td></td>
<td>where ( d = ) displacement relative to a remote point distance ( c ) from edge of loaded area</td>
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<td>At any point ( Q ) ( \sigma_c = \frac{p}{\ell} (\alpha + \sin \alpha) )</td>
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<td><strong>4d.</strong></td>
<td>Rigid cylindrical die of radius ( r ) on surface of semi-infinite body</td>
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<td>( d = F(1 - v^2)/2RE ) ( F ) at any point ( Q ) on surface of contact ( \sigma_c = \frac{F}{2\pi R \sqrt{R^2 - r^2}} )</td>
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<tr>
<td></td>
<td>( (\sigma_c)<em>{\max} = \infty ) at edge ( (\sigma_c)</em>{\min} = F/2\pi R^2 ) at center</td>
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<td><img src="image" alt="Diagram" /></td>
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</table>
4e. Uniform pressure $p$ over circular area of radius $R$ on surface of semi-infinite body

$$d_{\text{max}} = \frac{2pR(1-\nu^2)}{E} \text{ at center, } \quad d = \frac{4pR(1-\nu^2)}{\pi E} \text{ on the circle}$$

$$\tau_{\text{max}} = \frac{p}{2} \left[ \frac{1-2\nu}{2} + \frac{2}{9} (1 + \nu) \sqrt{2(1 + \nu)} \right]$$

at point $R \sqrt{2(1 + \nu)/(7 - 2\nu)}$ below center of loaded area

4f. Uniform pressure $p$ over square area of sides $2b$ on surface of semi-infinite body

$$d_{\text{max}} = \frac{2.24pb(1-\nu^2)}{E} \text{ at center, } \quad d = \frac{1.12pb(1-\nu^2)}{E} \text{ at corners}$$

$$d_{\text{ave}} = \frac{1.90pb(1-\nu^2)}{E}$$

---

$^a$ All diameters and radii are positive in formulas given. Values of $n_a, n_b, n_c,$ and $n_d$ are given in Table 9-3. Most of these formulas are adapted from Ref. [9.20].
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### TABLE 9-4  EQUATIONS FOR THE PARAMETERS OF TABLE 9-3

#### Definitions

\[
E(e) = \int_{0}^{\pi/2} \sqrt{1 - e^2 \sin^2 \phi} \, d\phi \\
K(e) = \int_{0}^{\pi} \frac{d\phi}{\sqrt{1 - e^2 \sin^2 \phi}} \\
e = \sqrt{1 - (b/a)^2} \quad k = b/a
\]

#### Equations

\[
\frac{A}{B} = \frac{K(e) - E(e)}{(1/k^2)E(e) - K(e)} \\
n_d = \frac{1}{k} \left( \frac{2kE(e)}{\pi} \right)^{1/3} \\
n_b = \left( \frac{2kE(e)}{\pi} \right)^{1/3} \\
n_c = \frac{1}{E(e)} \left( \frac{\pi^2kE(e)}{4} \right)^{1/3} \\
n_d = \frac{K(e)}{E(e)^{1/2}} \left( \frac{2k}{\pi} \right)^{2/3}
\]

\(^{a}\)Elliptic integrals \(E(e)\) and \(K(e)\) are tabulated and readily available in mathematical handbooks. Quantities \(a\) and \(b\) are semimajor and semiminor axes of the contact ellipse, respectively.
TABLE 9-5 CONTACT STRESSES BETWEEN TWO LONG CYLINDRICAL BODIES IN LINE CONTACT SLIDING AGAINST EACH OTHER<sup>a</sup>

**Notation**

- \(2b\) = width of contact area, Eq. (9.18a)
- \(\mu\) = friction coefficient
- \(\Delta\) = see Eq. (9.18b)
- \(q\) = distributed (line) load \((F/L)\)

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<th>Maximum Stress</th>
<th>(\mu = 0)</th>
<th>(\mu = \frac{1}{12})</th>
<th>(\mu = \frac{1}{6})</th>
<th>(\mu = \frac{1}{3})</th>
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<td>Maximum tensile principal stress, occurs near the surface at (x = -b)</td>
<td>0</td>
<td>(\frac{2}{12} \frac{b}{\Delta})</td>
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<td>(\frac{2}{3} \frac{b}{\Delta})</td>
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<td>Maximum compressive principal stress, occurs near the surface between (x = 0) and (x = 0.3b)</td>
<td>(-\frac{b}{\Delta})</td>
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<td>Maximum shear stress (occurs at the surface for (\mu \geq \frac{1}{10}))</td>
<td>0.300 (\frac{b}{\Delta})</td>
<td>0.308 (\frac{b}{\Delta})</td>
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<td>0.435 (\frac{b}{\Delta})</td>
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<td>Maximum octahedral shear stress (occurs at the surface for (\mu \geq \frac{1}{10}))</td>
<td>0.272 (\frac{b}{\Delta})</td>
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<td>0.277 (\frac{b}{\Delta})</td>
<td>0.368 (\frac{b}{\Delta})</td>
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<sup>a</sup>Adapted from [9.9], with permission.

---

TABLE 9-5 Contact Stresses between Two Long Cylindrical Bodies 449