Consider a three dimensional body occupying the volume $\Omega$, inside which heat is generated at a rate $G$ and whose outer surface $\Gamma$ is subjected to three distinct boundary conditions on portions $S_1$, $S_2$ and $S_3$. In rectangular Cartesian coordinates the governing equation is

$$\nabla \cdot (k \nabla T) + G = \frac{\partial}{\partial x} (k \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y} (k \frac{\partial T}{\partial y}) + \frac{\partial}{\partial z} (k \frac{\partial T}{\partial z}) + G = 0$$

and the boundary conditions are

$$T = T_b$$
on $S_1$,

$$k \nabla T \cdot \mathbf{n} + q = 0$$
on $S_2$, and

$$k \nabla T \cdot \mathbf{n} + h(T - T_\infty) = 0$$
on $S_3$.

The variational statement of the above problem involves the minimization of the functional

$$I(T) = \frac{1}{2} \int_{\Omega} [(\frac{\partial T}{\partial x})^2 + k(\frac{\partial T}{\partial y})^2 + k(\frac{\partial T}{\partial z})^2 - 2GT] d\Omega + \int_{S_2} qT ds + \frac{1}{2} \int_{S_3} h(T - T_\infty)^2 ds$$

In the finite element method the domain $\Omega$ is first subdivided into a number of regions with simple polygonal shape and the temperature within each element $T^e$ is expressed as a linear combination of the nodal values $T_i; i = 1, 2, ..., r$ by means of finite element basis functions, i.e.

$$T^e = \sum_{i=1}^{r} \phi_i T_i = \phi^T T$$
where $\phi$ is the basis function matrix and $T$ is the vector of nodal temperatures. In the Ritz formulation of the finite element method one looks for the values of $T_i$ that minimize $I(T) \approx \sum_{e=1}^{N} I^e(T^e)$, i.e.

$$\frac{\partial I}{\partial T_i} = \sum_{e=1}^{N} \frac{\partial I^e}{\partial T_i} = 0$$

for $i = 1, 2, \ldots, M$ where $M$ is the total number of nodes in the finite element mesh and $N$ is the total number of elements. The result is a set of $M$ linear algebraic equations for the $M$ unknown nodal values; the finite element equation

$$KT = F$$

where the stiffness matrix is given by

$$K = \int_{\Omega} B^T DB d\Omega + \int_{S_3} h\phi^T \phi ds$$

where

$$B = \begin{bmatrix} \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_2}{\partial x} & \ldots & \frac{\partial \phi_r}{\partial x} \\ \frac{\partial \phi_1}{\partial y} & \frac{\partial \phi_2}{\partial y} & \ldots & \frac{\partial \phi_r}{\partial y} \\ \frac{\partial \phi_1}{\partial z} & \frac{\partial \phi_2}{\partial z} & \ldots & \frac{\partial \phi_r}{\partial z} \end{bmatrix}$$

and

$$D = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

and the force vector is

$$F = \int_{\Omega} G\phi^T d\Omega - \int_{S_2} q\phi^T ds + \int_{S_3} kT_{\infty}\phi^T ds$$

Note that for insulated boundaries (i.e. $h = q = 0$) the corresponding integrals vanish and no term appears reflecting this contribution in the finite element equation (natural boundary condition).

In order to assure convergence of the finite element model as the element size decreases the basis functions must satisfy the standard requirements of compatibility and completeness, i.e. the basis functions selected must provide for continuity of the temperature at the interface between any two elements as well as for the continuity of temperature and heat flux inside each element. Standard polynomials of order $n$ satisfy the requirement.
Exercise 1
A fin is a rectangular prism with section 3 by 2 mm and is 2 cm long and is made of a material with \( k = 200 \text{W/mC} \). At its base, the temperature is maintained at 100 degrees Celsius while the other end is insulated. Heat is lost from the side surface of the fin by convection \((h = 120 \text{W/m}^2\text{C})\) into air at 25 degrees. Construct a finite element model of this system. Start with a 1D single element representation, then 2 elements, then 2D and finally 3D.

Exercise 2
A plane wall of thickness 60 mm and \( k = 21 \text{W/mC} \) has an internal source of heat releasing energy at a rate 0.3 \( \text{MW/m}^3 \) while the temperature at its surface is maintained at 40 degrees Celsius. Develop a finite element model and compare the results of the computation against the exact solution. Again, use 1D, 2D and 3D elements of various types and investigate convergence behavior.

Exercise 3
Same situation as in Exercise 2 but now the left face of the wall is subjected to convective heat transfer \((h = 570 \text{W/m}^2\text{C})\) from a medium at 93 degrees Celsius. Develop a finite element model to determine an approximation to the temperature distribution in the wall.

Exercise 4
A square plate of size 1 m by 1 m is insulated over its extended surface so that heat flows into and out of the plate through its edges. Three edges are maintained at 100 degrees Celsius and the fourth one at 500 degrees Celsius. The thermal conductivity of the material of the plate is \( k = 10 \text{W/mC} \). Develop a finite element model to determine an approximation to the temperature of the plate and compare your results against the exact solution.