Week 7
The Ritz-Galerkin Method II

1 Exercise 1

Consider in $x \in [0, 1]$, the second order Boundary Value Problem
\[
\frac{d^2 u}{dx^2} + u + x = 0
\]
subject to
\[u(0) = u(1) = 0\]

Construct the associated functional and find an approximate solution using the Ritz method. Construct also the variational statement of the problem and find an approximate solution using the Galerkin method. Assume a coordinate function of the form
\[\phi_1 = x(1 - x)\]
and an approximation of the form
\[u_1 = a_1 \phi_1 = a_1 x(1 - x)\]

2 Exercise 2

Consider in $x \in [0, 1]$, the second order Boundary Value Problem determined by the differential equation
\[
\frac{d^2 u}{dx^2} = -\frac{1}{1+x}
\]
and the boundary conditions
\[u(0) = u'(1) = 0\]
Construct the associated functional and find an approximate solution using the Ritz method. Construct also the variational statement of the problem and find an approximate solution using the Galerkin method. Assume approximations of the form

\[ u_n = a_1 x + a_2 x^2 + \ldots + a_n x^n \]

and compute in turn for \( n = 1, 2, 3 \)

### Exercise 3

Consider in \( x \in [0, L] \) the fourth order Boundary Value Problem determined by the differential equation

\[ [(x + 2L)u''']' + qu - kx = 0 \]

and the boundary conditions

\[ (x + 2L)u''' = 0 \]
\[ \frac{d}{dx}[(x + 2L)u''] = 0 \]

on \( x = 0 \), and

\[ u(L) = u'(L) = 0 \]

on \( x = L \).

Construct the associated functional and find an approximate solution using the Ritz method. Construct also the variational statement of the problem and find an approximate solution using the Galerkin method.

Assume coordinate functions of the form

\[ \phi_1 = (x - L)^2(x^2 + 2Lx + 3L^2) \]

and

\[ \phi_2 = (x - L)^3(3x^2 + 4Lx + 3L^2) \]

And assume in turn approximations of the form

\[ u_1 = a_1 \phi_1 \]

and

\[ u_2 = a_1 \phi_1 + a_2 \phi_2 \]
4 Exercise 4

Consider the BVP consisting of finding \( u(x) \) satisfying
\[
-\frac{d^2 u}{dx^2} = -u'' = \sin(\pi x)
\]
subject to
\[
\frac{du}{dx} = \frac{1}{\pi}
\]
at \( x = 0 \) and
\[
\frac{du}{dx} = -10u
\]
at \( x = 1 \).

a) Use the variational statement of the problem and find an approximate solution by assuming the functions \( u \) and \( v \) are both quadratic polynomials with the following forms
\[
u_3 = a_1 + a_2 x + a_3 x^2
\]
v_3 = b_1 + b_2 x + b_3 x^2

b) Repeat the calculation but now assuming the functions have instead the following forms
\[
u_3 = a_1 + a_2 x + a_3 x(1 - x)
\]
v_3 = b_1 + b_2 x + b_3 x(1 - x)

5 Exercise 5

Consider in \( x \in [-a, +a] \) and \( y \in [-b, b] \) the Boundary Value Problem
\[
u_{xx} + v_{yy} = -2
\]
giving the stress function for the torsion of a shaft with rectangular cross section.

Construct the associated functional and find an approximate solution using the Ritz method. Assume an approximation of the form
\[
u_1 = a_1 (a^2 - x^2)(b^2 - y^2)
\]
Assuming \( a = b = 1 \) compute the value of the torsional rigidity
\[
D = 2 \int \int u dxdy
\]
6 Exercise 6

Consider the problem of torsion of a hollow shaft analyzed in the Appendix of Courant’s paper. Construct the associated functional and find an approximate solution using the Ritz method. Compute the quantity $S$ and compare your results with those reported in the paper.