A Portfolio of Three Risky Assets

Not a two risky asset world
We need to be able to deal with portfolios that have many assets.

The return on a three-asset portfolio

\[ r_p = w_a r_a + w_b r_b + w_c r_c \]

- \( r_a \) is the return and \( w_a \) is the weight on asset A.
- \( r_b \) is the return and \( w_b \) is the weight on asset B.
- \( r_c \) is the return and \( w_c \) is the weight on asset C.
- \( w_a + w_b + w_c = 1 \)
A Portfolio of Three Risky Assets

Three asset portfolio: \( E(r_p) \) and \( \sigma(r_p) \)

If A, B, and C are assets, \( w_a, w_b, \) and \( w_c \) are the portfolio weights on each asset, and \( w_a + w_b + w_c = 1 \), then the expected return of the portfolio is

\[
E(r_p) = w_a E(r_a) + w_b E(r_b) + w_c E(r_c)
\]

the variance of the portfolio is

\[
\sigma^2(r_p) = w_a^2 \sigma^2(r_a) + w_b^2 \sigma^2(r_b) + w_c^2 \sigma^2(r_c)
\]
\[
+ 2w_a w_b \text{cov}(r_a, r_b) + 2w_a w_c \text{cov}(r_a, r_c)
\]
\[
+ 2w_b w_c \text{cov}(r_b, r_c),
\]

and the standard deviation of the portfolio is

\[
\sigma(r_p) = \sqrt{\sigma^2(r_p)}.
\]

A Portfolio of Three Risky Assets

Expected returns

The expected return of a portfolio is just a weighted sum of the individual security expected returns.

Variance is a bit more complicated

- Covariances are more influential than with a 2-asset portfolio.
- Three covariances now instead of one.
- As we add more assets, covariances become a more important determinate of the portfolio's variance (diversification).
Example: Three Risky Assets

You hate Microsoft so you invest in three competitors

<table>
<thead>
<tr>
<th></th>
<th>$E(r_i)$</th>
<th>Apple</th>
<th>Sun Micro</th>
<th>Red Hat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apple</td>
<td>0.20</td>
<td>0.09</td>
<td>0.045</td>
<td>0.05</td>
</tr>
<tr>
<td>Sun Micro</td>
<td>0.12</td>
<td>0.045</td>
<td>0.07</td>
<td>0.04</td>
</tr>
<tr>
<td>Red Hat</td>
<td>0.15</td>
<td>0.05</td>
<td>0.04</td>
<td>0.06</td>
</tr>
</tbody>
</table>

The covariance matrix

The diagonal of the covariance matrix holds variances.

$$\text{cov}(r_{\text{apple}}, r_{\text{apple}}) = \sigma^2(r_{\text{apple}})$$

The covariance matrix is symmetric (row 1, column 2 = row 2, column 1):

$$\text{cov}(r_{\text{apple}}, r_{\text{sun}}) = \text{cov}(r_{\text{sun}}, r_{\text{apple}})$$

Example: Three Risky Assets

Computing $E(r_p)$ and $\sigma(r_p)$

$$E(r_p) = w_aE(r_a) + w_sE(r_s) + w_hE(r_h)$$
$$= \frac{1}{3}(0.2) + \frac{1}{3}(0.12) + \frac{1}{3}(0.15) = 15.7\%$$

$$\sigma^2(r_p) = w^2_a\sigma^2(r_a) + w^2_s\sigma^2(r_s) + w^2_h\sigma^2(r_h)$$
$$+ 2w_aw_s \text{cov}(r_a, r_s) + 2w_aw_h \text{cov}(r_a, r_h)$$
$$+ 2w_sw_h \text{cov}(r_s, r_h)$$
$$= \frac{1}{9}(0.09) + \frac{1}{9}(0.07) + \frac{1}{9}(0.06)$$
$$+ 2(\frac{1}{3}\frac{1}{3})(0.045) + 2(\frac{1}{3}\frac{1}{3})(0.05) + 2(\frac{1}{3}\frac{1}{3})(0.04)$$
$$= 0.0544$$

$$\sigma(r_p) = \sqrt{0.0544} = 23.3\%$$
A Portfolio of Many Risky Assets

An N asset portfolio

Consider a portfolio that has \( n \) assets (\( n \) can be a very large number like 1000, 7000, or even more than that).

The return on an N asset portfolio

\[
 r_p = w_1 r_1 + w_2 r_2 + \cdots + w_n r_n \\
= \sum_{i=1}^{n} w_i r_i
\]

- \( r_i \) is the return on the \( i \)th asset in the portfolio.
- \( w_i \) is the portfolio weight for the \( i \)th asset.
- The sum of the weights equals one: \( \sum_{i=1}^{n} w_i = 1 \)

Many Risky Assets: Expected Return

Expected return still a weighted sum

The expected return is a weighted average of the expected returns of the individual securities in the portfolio:

\[
 E(r_p) = E\left(w_1 r_1 + w_2 r_2 + \cdots + w_n r_n\right) \\
= w_1 E(r_1) + w_2 E(r_2) + \cdots + w_n E(r_n) \\
= \sum_{i=1}^{n} w_i E(r_i)
\]
Many Risky Assets: Variance

The variance

\[
\sigma^2_{r_p} = w_1^2 \sigma^2(r_1) + w_2^2 \sigma^2(r_2) + \cdots + w_n^2 \sigma^2(r_n) \\
+ 2w_1w_2 \text{cov}(r_1, r_2) + 2w_1w_3 \text{cov}(r_1, r_3) + \cdots + 2w_1w_n \text{cov}(r_1, r_n) \\
+ 2w_2w_3 \text{cov}(r_2, r_3) + 2w_2w_4 \text{cov}(r_2, r_4) + \cdots + 2w_2w_n \text{cov}(r_2, r_n) \\
: \\
+ 2w_{n-2}w_{n-1} \text{cov}(r_{n-2}, r_{n-1}) + 2w_{n-2}w_n \text{cov}(r_{n-2}, r_n) \\
+ 2w_{n-1}w_n \text{cov}(r_{n-1}, r_n)
\]

Writing the variance more compactly

\[
\sigma^2(r_p) = \sum_{i=1}^{n} w_i^2 \sigma^2(r_i) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} 2w_iw_j \text{cov}(r_i, r_j)
\]

Thinking About Risk

Risky Securities
What makes a stock risky?

Risky Portfolios
What makes a portfolio risky?
Many Risky Assets: Variance

The importance of covariances

- If the number of securities is large (and the weights are small), covariances become the most important determinant of a portfolio’s variance.
- For a $n$ asset portfolio there are $n(n - 1)/2$ covariance terms, and $n$ variance terms.
- For example, a portfolio of a 1000 stocks, has 1000 variance terms and 499,950 unique covariance terms.
- Compare that to a portfolio of two assets which has 2 variance terms, and 1 covariance term.

The Power of One

How does an asset affect a portfolio’s variance?

Just find all the terms in the variance equation that involve the asset. Let’s do it for asset 1.

$$w_1^2 \sigma^2(r_1) + 2w_1w_2 \text{cov}(r_1, r_2) + \cdots + 2w_1w_n \text{cov}(r_1, r_n)$$

The risk of security 1 in portfolio $p$

- An asset’s influence on a portfolio’s variance primarily depends on how it covaries with the other assets in the portfolio.
- Thus, what matters is how an asset covaries with the portfolio. The above expression almost equals,

$$w_1 \text{cov}(r_1, r_p)$$
Minimizing the Portfolio Variance

A small thought experiment
To illustrate the importance of covariances let’s think about how we could find the minimum variance portfolio.

Finding the portfolio with the smallest possible variance

1. Find two securities already in the portfolio with different covariances with the portfolio.
2. Add a little weight to the security with a lower $\text{cov}(r_i, r_p)$, and subtract a little from the security with the higher covariance.
3. The portfolio variance is a little lower. Repeat steps 1 and 2 until the variance cannot be lowered anymore.

The variance of the portfolio will be minimized when all the securities have the same covariance with the portfolio:

$$\text{cov}(r_1, r_p) = \text{cov}(r_2, r_p) = \cdots = \text{cov}(r_n, r_p)$$

Mean Variance Analysis

Asset allocation with many assets

- Suppose you can invest in many risky assets. What is the optimal allocation?
- Clearly it depends on preferences.
- Can we say anything about the portfolios that should be chosen?
- What property will the optimal or best portfolios have?
- What does the investment opportunity set look like when we have many assets?
The Investment Opportunity Set

The investment opportunity set

When there are many risky assets the feasible investment set is a curve and the area to the right of the curve.

The boundary and the frontier

- The curve in the graph is called the mean-variance boundary or the mean-variance frontier.
- The portion of the curve above the global minimum variance portfolio is called the mean-variance efficient frontier or just the efficient frontier.
- An investor will only choose a portfolio on the efficient frontier.
- Portfolios on the efficient frontier are called mean variance efficient portfolios or efficient portfolios.
- Adding more assets pushes the curve towards the Northwest corner of the graph.
The Mean Variance Boundary

How is the mean-variance boundary formed?
We pick the expected return we want, and then choose the weights of the portfolio so that the variance of the portfolio is minimized.
## Two-Fund Separation

### Two-Fund Separation
- The entire mean-variance boundary can be created from any two portfolios on the mean-variance boundary.
- The entire efficient frontier can be created from any two mean variance efficient portfolios.

### More implications
- A portfolio of portfolios on the mean-variance boundary will also be on the boundary.
- A portfolio of mean variance efficient portfolios will be on the efficient frontier if the weight on each portfolio is positive.

## Adding a Riskless Asset

- How do you allocate your money between a riskless asset and risky mean variance efficient portfolios?
- Logically, this is the same as when we only had two risky assets.
- We can draw a CAL between every efficient portfolio (the only ones we care about) and the riskfree asset.
- The best or optimal CAL is the one with the highest slope.
- The optimal risky portfolio is the portfolio that is tangent to the optimal CAL (the tangency portfolio).
### Risk Free Asset and Many Risky Assets

The Optimal CAL

- **E(r)**: Expected return
- **σ(r)**: Standard deviation
- **CAL 1**, **CAL 2**, **CAL 3**: Capital Allocation Line
- **Risk Free Asset**: The tangency portfolio
The Optimal CAL

The optimal CAL: Definition
The CAL with the highest slope (Sharpe ratio):

\[
\text{Slope} = \frac{E(r_p) - r_f}{\sigma(r_p)}
\]

The tangency portfolio
- The optimal CAL is the tangency line between the riskfree rate and the mean variance efficient frontier comprised entirely of risky assets.
- The tangency portfolio is the portfolio – comprised of only risky assets – with the highest Sharpe ratio.

The New Efficient Frontier

The new efficient frontier is the optimal CAL
If there are many risky assets and a riskfree asset, the efficient frontier is the optimal CAL.

Optimal Allocations
- Investors combine the tangency portfolio with the riskfree asset to form their overall portfolio.
- The allocation they choose depends on their preferences for risk and expected return.
- Note: Two-fund separation still holds; we can trace out the entire frontier with two portfolios on the frontier.
  - For example, the tangency portfolio and the riskfree asset.
Finding the Tangency Portfolio

How do we find the tangency portfolio?
It is the portfolio – comprised of only risky assets – with the highest Sharpe ratio:

$$SR = \frac{E(r_p) - r_f}{\sigma(r_p)}$$

An incorrect method

1. Form a portfolio using all the risky securities (any portfolio will do).
2. Find two securities already in the portfolio with different Sharpe ratios.

Wrong! Step 2 is not quite right. Let’s think about the goal.
Finding the Tangency Portfolio

What we care about
- We only care about how a security affects the portfolio’s Sharpe ratio.
- Remember, securities only affect the variance of a portfolio via its covariance with the portfolio.

Examine the marginal sharpe ratios
- We really just need to look for securities with different,
  \[
  \frac{E(r_i) - r_f}{\text{cov}(r_i, r_p)}
  \]
- Note: the above ratio is called the **marginal Sharpe ratio**.

The correct method
1. Form a portfolio using all the risky securities (any portfolio will work).
2. Find two securities already in the portfolio with different risk premium to covariance ratios:
   \[
   \frac{E(r_i) - r_f}{\text{cov}(r_i, r_p)}
   \]
3. Add a little weight to the security with a higher ratio, and subtract a little from the security with the lower ratio.
4. Keep repeating steps 1 – 3 until, marginal Sharpe ratios are equal across all securities.
Finding the Tangency Portfolio

All the marginal Sharpe ratios are equal

- For the tangency portfolio (T), the following is true:

\[
\frac{E(r_1) - r_f}{\text{cov}(r_1, r_T)} = \frac{E(r_2) - r_f}{\text{cov}(r_2, r_T)} = \cdots = \frac{E(r_n) - r_f}{\text{cov}(r_n, r_T)}
\]

- This includes all individual securities and other portfolios.

Rewriting compactly

We can write the tangency condition as the following:

\[
\frac{E(r_i) - r_f}{\text{cov}(r_i, r_T)} = \frac{E(r_j) - r_f}{\text{cov}(r_j, r_T)} \quad \text{for all } i \text{ and } j
\]

Example: The Tangency Portfolio

Suppose you can invest in two risky assets and a riskfree asset

<table>
<thead>
<tr>
<th></th>
<th>E(r)</th>
<th>(\sigma)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock 1</td>
<td>11%</td>
<td>20%</td>
</tr>
<tr>
<td>Stock 2</td>
<td>16%</td>
<td>25%</td>
</tr>
<tr>
<td>Riskfree</td>
<td>6%</td>
<td></td>
</tr>
</tbody>
</table>

\[
\text{cov}(r_1, r_2) = 0.025
\]

The tangency condition

Remember, the following is true for the tangency portfolio (T):

\[
\frac{E(r_1) - r_f}{\text{cov}(r_1, r_T)} = \frac{E(r_2) - r_f}{\text{cov}(r_2, r_T)}
\]
Example: The Tangency Portfolio

Computing covariance of a security with a portfolio

If \( x, y, \) and \( z \) are random variables and \( A \) and \( B \) are constants,

\[
\text{cov}(z, Ax + By) = A \text{cov}(z, x) + B \text{cov}(z, y)
\]

If portfolio \( P \) is comprised of two assets called \( A \) and \( B \), \( w \) is the weight on \( A \) and \( 1 - w \) is the weight on \( B \), then the covariance of a security \( I \) with portfolio \( P \) is the following:

\[
\text{cov}(r_i, r_p) = w \text{cov}(r_i, r_a) + (1 - w) \text{cov}(r_i, r_b)
\]

The rule applied to the example

\[
\text{cov}(r_1, r_T) = w \text{cov}(r_1, r_1) + (1 - w) \text{cov}(r_1, r_2)
= w \sigma^2(r_1) + (1 - w) \text{cov}(r_1, r_2)
\]

\[
\text{cov}(r_2, r_T) = w \text{cov}(r_2, r_1) + (1 - w) \text{cov}(r_2, r_2)
= w \text{cov}(r_1, r_2) + (1 - w) \sigma^2(r_2)
\]

Now we are ready to find the tangency portfolio

We can solve for the optimal \( w \) by using the tangency portfolio condition:

\[
\frac{E(r_1) - r_f}{\text{cov}(r_1, r_T)} = \frac{E(r_2) - r_f}{\text{cov}(r_2, r_T)}
\]
Example: The Tangency Portfolio

\[
\frac{E(r_1) - r_f}{\text{cov}(r_1, r_T)} = \frac{E(r_2) - r_f}{\text{cov}(r_2, r_T)}
\]

\[
(E(r_1) - r_f) \text{cov}(r_2, r_T) = (E(r_2) - r_f) \text{cov}(r_1, r_T)
\]

\[
5[\text{wcov}(r_1, r_2) + (1 - w)\sigma^2(r_2)] = 10[w\sigma^2(r_1) + (1 - w)\text{cov}(r_1, r_2)]
\]

\[
5[0.025w + (1 - w)0.0625] = 10[0.04w + (1 - w)0.025]
\]

\[
5[-0.0375w + 0.0625] = 10[0.015w + 0.025]
\]

\[
-0.1875w + 0.3125 = 0.15w + 0.25
\]

\[
0.3375w = 0.0625
\]

\[
w = 0.185
\]

The optimal portfolio is 18.5% in stock 1 and 81.5% in stock 2.

What is the Sharpe Ratio of the tangency portfolio?

\[
E(r_T) = 0.185(0.11) + 0.815(0.16) = 15.08\%
\]

\[
\sigma^2(r_t) = 0.185^2(0.2^2) + 0.815^2(0.25^2) + 2(0.185)(0.815)(0.025)
\]

\[
= 0.05
\]

\[
SR_T = \frac{0.1508 - 0.06}{\sqrt{0.05}} = 0.406
\]

Compare that to the Sharpe ratios of stock 1 and 2:

\[
SR_1 = \frac{0.11 - 0.06}{0.20} = 0.25
\]

\[
SR_2 = \frac{0.16 - 0.06}{0.25} = 0.40
\]
A Few Things to Think About

Problems with mean variance analysis

- Can you think of any limitations with respect to mean variance analysis?
- What are some of the practical problems people face when they try to use mean variance analysis?
- How might people manage some of these practical problems?

Warren Buffet

- Warren Buffet claims he doesn’t engage in mean variance analysis.
- Is that a mistake or can you think of reasons why it doesn’t make sense for him to use mean variance analysis?

Summary

- An asset’s influence on a portfolio’s variance primarily depends on how it covaries with the other assets in the portfolio.
- A rational and risk averse investor will only invest in efficient portfolios.
- Efficient portfolios have the highest expected return for a given standard deviation.
- If there is a risk free asset then the efficient frontier is a CAL line between the riskfree asset and the tangency portfolio.
- The tangency portfolio is composed of only risky assets.
- The tangency portfolio has the highest Sharpe ratio of any portfolio composed of risky assets.
- In the tangency portfolio all the marginal Sharpe ratios are equal.