Modal Analysis of Rectangular Simply-Supported Functional Graded Plates

by

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Approved:

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# NOMENCLATURE

- $\rho$  Density (kg/m³)
- $E$  Modulus of Elasticity (Pa)
- $\nu$  Poisson’s Ratio (-)
- $D$  Flexural Rigidity (Pa·m³)
- $h$  Thickness (m)
- $a$  Plate Length (m)
- $b$  Plate Length (m)
- $f$  Frequency (Hz)
- $V_c$  Volume Fraction, material 1 (dimensionless)
- $V_f$  Volume Fraction, material 2 (dimensionless)
- $V_{\text{bot}}$  Volume Fraction of material 1 at bottom of plate (dimensionless)
- $V_{\text{top}}$  Volume Fraction of material 1 at top of plate (dimensionless)
- $K$  Shear Modulus (Pa)
- $\mu$  Bulk Modulus (Pa)
- $n$  Power (dimensionless)
ACKNOWLEDGMENT

I would like to thank my family for continues support and inspiration during my education. I would also like to thank the faculty at RPI Hartford, especially Professor Ernesto Gutierrez-Miravete, for their guidance.
ABSTRACT

Functionally graded materials (FGM) are defined as an anisotropic material whose physical properties vary continuously throughout the volume, either randomly or strategically, to achieve desired characteristics or functionality [1]. The aim of this project is to perform a modal analysis to determine the natural frequencies and mode shapes of FGM using Finite Element Analysis (FEA). A modal analysis will be performed on isotropic cases with FEA and compared its known theoretical solution. FGMs are then chosen in increasing complexity and modal analysis is performed on each case. These analyses are compared to the natural frequencies of the constituent materials computed in the isotropic cases. The natural frequencies of the FGMs are compared known solutions in literature, if they exist or compared to analytic models for computing FGMs [2]. In all cases of modal analysis, the FEA are validated by mesh extension trials and convergence tests.
1. Introduction

1.1 Background

1.1.1 Functional Graded Materials
FGM are typically designed for a specific function or application. Many times they are manufactured to achieve good strength to weight ratios, good thermal or electrical conductivity, or various other material advantages. FGMs differ from traditional composites in that their material properties vary continuously, where the composite changes at each laminate interface. [1] FGMs accomplish this by gradually changing the volume fraction of the materials which make up the FGM. [1]

1.1.2 Modal Analysis
Many times when using FGM, or any material, in a structural or mechanical application, it often required to know the acoustic properties of the material. This project aims to discover the natural frequencies and accompanying mode shapes of the FGM, which are the key parameters when considering acoustic performance. To determine these parameters, modal analysis will be used. Modal analysis is a method of determining the natural acoustic characteristics, or dynamic response, of materials. The analysis involves imposing an excitation into the structure and finds the given frequency at which the structure resonates, (i.e. when the excitation and the vibration response match). A typical modal analysis will return multiple frequencies, each with an accompanying displacement field which the structure experiencing at that frequency. Each frequency is known and a “mode” and the displacement field is known as the “mode shape”.

1.2 Problem Description
This project aims to discover the natural frequencies and accompanying mode shapes of the FGM, which are the key parameters when considering acoustic performance. This project performs modal analysis on the following cases:
   A) Isotropic
a. Steel  
b. Aluminum  
c. Alumina  
d. Zirconia  

B) A linear FGMs  
a. Steel and aluminum  

C) A power law (n=2) FGMs  
a. Steel and alumina  
b. Aluminum and zirconia  

D) A power law (n=10) FGMs  
a. Steel and alumina  
b. Aluminum and zirconia  

Cases A-D are each 1m x 1m plates, with thicknesses of 0.025m and 0.05m, and have simply-supported boundary conditions. Each case four frequencies and mode shapes are extracted from the COMSOL software. The frequencies from Case A are compared to the exact solution of an isotropic plate given in Reference [3]. For Cases B-D, each material is assumed to be 100% and 0% at each face of the plate. To validate the models of Cases B-D, a modal analysis of the FGM with the volume fraction approaching each of the isotropic materials and a convergence check to Case A values is performed. Each case will be compared its constituent isotropic case to determine the contribution of each material to the frequency and mode shape. Each result for the FGMs are compared to equation 9 of Reference [2].

The cases were chosen so a variety of material conditions could be investigated. For instance, the n=10 power law case represents a material that is made up of one material for the large majority of the material, then has a thin layer of another material on the top. This case simulates what would be considered a metal with a thin ceramic coating.

To capture the effective material properties of the FGMs at a point in the thickness, the Mori-Tanaka (MT) estimation is used. [1] MT is dependent on the volume fractions of each constituent materials making up the FGM at a certain point.
For each material above, Table 1 contains their material properties of interest.

<table>
<thead>
<tr>
<th>Material</th>
<th>E [Pa]</th>
<th>ν</th>
<th>ρ [kg/m³]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel</td>
<td>1e11</td>
<td>.3</td>
<td>7800</td>
</tr>
<tr>
<td>Aluminum</td>
<td>7.5e9</td>
<td>.33</td>
<td>2700</td>
</tr>
<tr>
<td>Alumina</td>
<td>3e11</td>
<td>.27</td>
<td>3690</td>
</tr>
<tr>
<td>Zirconia</td>
<td>2e11</td>
<td>.3</td>
<td>5700</td>
</tr>
</tbody>
</table>
2. Methodology

2.1 Finite Element Modeling

COMSOL software was used in the modeling of the isotropic and FGM plates. Appendix A has details on the specific mesh selection that went into each model. It should be noted that different elements and mesh densities were used for isotropic and FGM plates.

To simulate the simply-supported boundary conditions, specified displacements were placed on the bottom edges and at one bottom corner. Edges 1-4 were constrained in the \( z \)-direction, and corner 5 was constrained in the \( x \), \( y \), and \( z \)-directions.

![Figure 1 – Simply-Supported Boundary Conditions](image)

2.2 Modal Analysis

The modal analysis is performed by using the eigenfrequency module, in the Solid Mechanics physics section of COMSOL. For each case, four frequencies and four mode shapes are computed. The mode shapes are plotted for each frequency over the geometry.

The exact solution for the isotropic case is taken from Timoshenko [3] and given as:

\[
 f_{mn} = \frac{\pi}{2} \sqrt{\frac{D}{\rho h \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) }}
\]
For a given FGM, the Mori-Tanaka Estimation [4] gives three material properties of interest in the modal analysis. The first and simplest is the density of the FGM, which follows the general mixture rule form [1]:

\[
D = \frac{E h^2}{12(1 - v^2)}
\]

2.3 Mori-Tanaka Estimation of FGM Properties

The last two items are the shear and bulk moduli, given by the following:

\[
\rho_{FGM} = \rho_M V_M + \rho_C V_C
\]

\[
K_{FGM} = K_M + \frac{(K_C - K_M) V_C}{1 + \left(1 - V_C\right)\left(K_C - K_M\right) K_M + \left(\frac{4}{3}\right) \mu_M}
\]

\[
\mu_{FGM} = \mu_M + \frac{(\mu_C - \mu_M) V_C}{1 + \left(1 - V_C\right)\left(\mu_C - \mu_M\right) \mu_M + f_1}
\]

\[
f_1 = \frac{\mu_M (9K_M + 8\mu_M)}{6(K_M + 2\mu_M)}
\]

Finally using elasticity, Young’s modulus and the Poisson ratio can be determined by inserting equations (3) thru (4a) into equations (5) and (6).

\[
E_{FGM} = \frac{9K_{FGM}\mu_{FGM}}{(\mu_{FGM} - 3K_{FGM})}
\]

\[
\nu_{FGM} = \frac{3K_{FGM} - 2\mu_{FGM}}{\mu_{FGM} + 6K_{FGM}}
\]

For Cases B-D, the \(V_c\) is a function of thickness of the plate. It assumed that the \(V_c\) and the \(V_m\) combine to unity and each material is at 100% at its respective plate face. Thus, \(V_{top} = 1\) and \(V_{bot} = 0\).
Table 2 – $V_c$ Equations for Different FGMs

<table>
<thead>
<tr>
<th>Case</th>
<th>$V_c$ Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>$V_c = \left(\frac{V_T - V_B}{h}\right)z + \left(\frac{V_T + V_B}{2}\right)$ (7)</td>
</tr>
<tr>
<td>Power Law, n=2</td>
<td>$V_c = V_B + (V_T - V_B)\left(\frac{1}{\sqrt{n}} + \frac{z}{h}\right)^2$ (8)</td>
</tr>
<tr>
<td>Power Law, n=10</td>
<td>$V_c = V_B + (V_T - V_B)\left(\frac{1}{\sqrt{n}} + \frac{z}{h}\right)^{10}$ (9)</td>
</tr>
</tbody>
</table>

Figure 3 – Graphical Representation of $V_c$ of Different FGMs Through Thickness of Specimen

With these relations of $V_c$ known, the Mori-Tanaka estimation can be used to predict material properties in Cases B-D.
3. Results and Discussion

3.1 Isotropic (Case A)

Table 3 – Isotropic Frequency Results

<table>
<thead>
<tr>
<th>Material</th>
<th>Mode</th>
<th>h=0.025</th>
<th>h=0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Frequency (Theory)</td>
<td>Frequency (Finite Element)</td>
</tr>
<tr>
<td>Steel</td>
<td>1</td>
<td>85.10</td>
<td>84.55</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>212.75</td>
<td>211.79</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>212.75</td>
<td>211.84</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>340.40</td>
<td>338.07</td>
</tr>
<tr>
<td>Aluminum</td>
<td>1</td>
<td>126.59</td>
<td>125.82</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>316.46</td>
<td>315.1</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>316.46</td>
<td>315.18</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>506.34</td>
<td>503.04</td>
</tr>
<tr>
<td>Alumina</td>
<td>1</td>
<td>212.32</td>
<td>210.88</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>530.79</td>
<td>528.32</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>530.79</td>
<td>528.44</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>849.26</td>
<td>707.43</td>
</tr>
<tr>
<td>Zirconia</td>
<td>1</td>
<td>140.78</td>
<td>139.88</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>351.96</td>
<td>350.37</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>351.96</td>
<td>350.46</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>563.14</td>
<td>463.4</td>
</tr>
</tbody>
</table>

The isotropic case is performed for 3 reasons: 1) To have a reliable case to test the modal analysis on FEA, 2) To have known isotropic results to use in comparison to the results for when given materials are used as constituents in FGM, and 3) To find a suitable thickness of a FGM plate so that error between theoretical values and FEA values is not too large (approx <5%).

In general, the isotropic case, both the 0.025m and the 0.05m plates, showed good agreement with their respective theoretical values. This validates the FEA approach used in COMSOL for further use on the FGM. The isotropic case for plate thickness 0.025 m had the two ceramic materials (Alumina and Zirconia) both experience a jump in error in the fourth mode. This can be attributed to mode 4 (typically) being a bending mode [2] and most thin ceramics are brittle and do not deform in the same
manner as a typical metal. It should be noted that the same spikes are not seen in the 0.05 m thick plate.

The 0.05m thick plate was found to be the thickest plate without the error of the isotropic frequencies exceeding approximately 5% of the theoretical value. This is significant because the FGM should be as thick as possible so that the functional dependency of the material properties through the thickness can be adequately captured. Also, the isotropic values are used for comparison to validate the results for the modal analysis of the FGM, so error should be kept as low as possible.

Figure 3 – Isotropic Mode Shape Results

The mode shapes shown in Figure 3 are obtained from the FEA modal analysis for each material. For each isotropic material, the mode shape is the same, but the scale of deformation varies for each material. For the purpose of this project, only the shape shall be considered. The results match the typical simply-supported rectangular plate shown in Reference [5].
3.2 Linear FGM

Table 4 – Linear FGM Frequency Results

<table>
<thead>
<tr>
<th>FGM</th>
<th>Mode</th>
<th>Frequency (Finite Element)</th>
<th>Efraim Formula [2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bottom Material: Steel</td>
<td>1</td>
<td>116.66</td>
<td></td>
</tr>
<tr>
<td>Top Material: Aluminum</td>
<td>2</td>
<td>291.81</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>291.83</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>340.40</td>
<td></td>
</tr>
</tbody>
</table>

Two initial checks of the veracity of the FGM frequency solution involve the isotropic solutions found in section 3.1. First, being that since the FGM is made of two known isotropic materials, one can safely assume that the frequencies that correspond to each material are the bounds of the FGM frequency solution. For the given linear FGM, the constituents are steel and aluminum. Consider the first mode, the first mode isotropic frequencies of steel and aluminum are 170.20 Hz and 80.06 Hz. Thus, 170.20 Hz is the upper bound and 80.06 Hz is the lower bound on the FGM first mode frequency. The solution of the frequency for the first mode of the linear FGM falls comfortably in these bounds. The second check would be to perform a quick calculation using the assumptions given by Efraim [2]. Efraim [2] states that the volume fractions at the top ($V_{\text{top}}$) and the bottom ($V_{\text{bot}}$) can be approximated by integrating the functional dependence of the FGM over thickness of the plate, and achieving the simple formulas for a power law:

\begin{align}
V_{\text{bot}} &= \frac{n}{1 + n} \\
V_{\text{top}} &= \frac{1}{1 + n}
\end{align}

For a linear case ($n=1$), the $V_{\text{top}}$ and $V_{\text{bot}}$ both would be 0.5. Since the FGM is split equally between steel and aluminum, another way to check the solution would be to compare it to the arithmetic average of the isotropic frequencies. The average is about 125 Hz, so the solution for the first mode is close.
The mode shapes for mode 2 and mode 3 have swapped from where they were in the isotropic cases.
4. Conclusions
5. References