Rensselaer Polytechnic Institute
Department of Engineering

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MANE 6980: Mechanical Engineering Project

A Study of the use of Perturbation Methods to Approximate Solutions to Non-linear Oscillatory Problems

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Student: William Royle
Advisor: Professor Ernesto Gutierrez-Miravete
Rensselaer; Hartford
Abstract

The overview of this project is to learn and apply perturbation, in order to approximate solutions to engineering problems which would otherwise be intractable through the use of traditional analytical methods. The project will first outline the technique of perturbation theory with the aid of algebraic equations. Approximate solutions for several variations of a non-linear mass spring dampener problems using various perturbation methods will be determined. The following is a list of the problems to be investigated over the duration of the project: Duffing’s equation, the van der Pol equation, a grounded mass connected to a linear and non-linear spring, and finally the chatter in metal turning. Traditional analytical, or numerical as required, solutions to these problems will be found and compared and contrasted to the approximations found through the use of perturbation theory.
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**Introduction**

Perturbation methods, also known as asymptotic, allow the simplification of a complex mathematical problem. Use of perturbations theory will allow approximate solutions to be determined for problems which cannot be solved by traditional analytical methods. Second order ordinary linear differential equations can be solved by engineers and scientists routinely. However in many cases, real life situations can require much more difficult mathematical models, such as a non-linear differential equation.

Numerical methods used on a computer of today are capable of solving extremely complex mathematical problems, they are not perfect. Numerical methods of today can still run into a multitude of problems ranging for diverging solutions to tracking wrong solutions. Numerical methods on a computer do not provide much insight to the engineers or scientists running them. Perturbation theory can offer an alternative approach to solving certain types of problems. Solving problems analytically often help an engineer or scientist to understand at a deeper level a physical problem and may help improve future procedures used to solve their problems. Since Also, in a time where there are tough economic circumstances, it is not unreasonable to consider that future employers may prefer to rely on human ingenuity over the necessity of continually purchasing expensive software package licenses.

The first step required to start the implementation of perturbation theory starts in nondimensionalizing the governing equation. Once the equation is non-dimensionalized, perturbation theory requires taking advantage of a “small” parameter that appears in an equation. This parameter, usually denoted “ε”, is on the order of $0 < \varepsilon < 1$.

Next, through educated assumptions on the order of magnitude of terms, a rough approximate solution is determined through the use of logical elimination of low impacting terms. The perturbation method than solves this reduced “outer problem”. Next an “inner solution” is constructed to satisfy the other constraints of the problem. A composite solution is obtained through a matching process.

Once a rough approximate solution is found, a “correction factor” may then be determined using a similar method of educated order of magnitude analysis. While “correction factors” can be used repeatedly, it is important to note, only a limited accuracy can be obtained through perturbation theory. Excess correction terms will eventually result in diverging off the actually solution. This is unlike a typical series solution, in which every additional term improves accuracy of the solution.

To help understand conceptually the mechanics of perturbation, the following example commonly known to most graduate level students is utilized. The equation of continuity in Cartesian Coordinates is as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$
The Navier Stokes equations for a Newtonian fluid with constant density and viscosity in Cartesian coordinates is as follows:

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] + \rho g_x
\]

\[
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right] + \rho g_y
\]

\[
\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right] + \rho g_z
\]

Assuming a steady, constant density ad viscosity, two dimensional flow, the continuity and Navier stokes equations reduce to the following:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]
\]

\[
\rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right]
\]

0 = 0

These equations are often used to model flow in boundary layer region. Often times, these equations are further simplified by Engineers and scientist depending on the physics of the problem being solved. This simplification can be performed by an order of magnitude analysis. For example, the velocity in the vertical plane may be extremely small compared to the velocity in the horizontal direction, therefore terms that carry the vertical velocity term will be reduced to zero. While the vertical velocity may not be zero exactly, this assumption will introduce some error into an eventual solution. The problem can be continued to be simplified in this manor until an analytical solution is often obtainable. The mechanics of perturbation theory follows this same manor, allowing analytical approximations to be found for equations which would otherwise be impossible to solve without the use of a computer.
**Problem Description**

This project objective is to study, learn and introduce the perturbation method with the support of simple algebraic equations. Once the perturbation method is introduced, it will be used to develop a set of approximate solutions for the following differential equation variations: Duffing’s equation, the van der Pol equation, a grounded mass connected to a linear and non-linear spring, and finally the chatter in metal turning.

The perturbation solution graphs can be compared to analytical or numerical solutions graphs obtained to the same problems throughout the study. This will allow for not only confirmation of the correct application of the perturbation method, but will also allow for the solutions to be compared and contrasted.

The following is a list of the problems to be solved throughout the duration of the project:

**Algebraic Equation**

\[ x^2 + \varepsilon x - 1 = 0 \]

**Linear Ordinary Differential Equation**

\[ \varepsilon \frac{d^2x}{dt^2} + 2 \frac{dx}{dt} + 2x = 0 \]

**Unforced Duffing Equation**

\[ \frac{d^2x}{dt^2} + x + \varepsilon \alpha x^3 = 0 \]

Where \( \alpha \) is considered to be a constant.

**Forced Duffing Equation**

\[ \frac{d^2x}{dt^2} + x + \varepsilon \alpha x^3 = F_0 \sin(\omega t) \]

Where \( \alpha \), \( F_0 \), and \( \omega \) are considered to be constants.

**Van Der Pol Equation**

\[ \frac{d^2x}{dt^2} + x + \varepsilon (x^2 - 1) \left( \frac{dx}{dt} \right) = 0 \]
Grounded Mass Connected to a Linear and Non-Linear Spring in Series

\[
\frac{d^2 x}{dt^2} + \omega^2 x + \varepsilon \left[ 3\eta x^2 \frac{d^2 x}{dt^2} + 6\eta x \left( \frac{dx}{dt} \right)^2 + \omega^2 x^3 \right] = 0
\]

Where \( \omega \) and \( \eta \) are consider to be constants.

Chatter in Metal Turning

\[
\frac{d^2 x}{dt^2} + 2\zeta \frac{dx}{dt} + p^2 (x + \beta_2 x^2 + \beta_3 x^3) = -p^2 w [x - x_\tau + \alpha_2 (x - x_\tau)^2 + \alpha_3 (x - x_\tau)^3]
\]

For;

\[ x_\tau = x(t - \tau) \]

Where \( p, \zeta, \beta_2, \beta_3, \alpha_2, \alpha_3 \) and \( w \) are consider to be constants.
Methodology

A polynomial algebraic equation will be solved using the traditional quadratic formal. Next, solutions for the same equation will be approximated following the techniques of perturbation theory. This will be done to develop the understanding of the methodology required. Next, the Poincare-Lindstedt method of determining approximate solutions will be applied to the governing response differential equations for the variety of problems identified.

Analytical solutions can be found in most instances using tradition methods for solving ordinary differential equations; however numerical solutions, such as runge-kutta, will be required to find solutions to the non-linear differential equations.

Microsoft Excel will be used to graph and compare analytical solutions to the approximate solutions obtained through the use of perturbation theory. MAPLE will be used to find numerical solutions as needed.
**Required Resources**

To achieve successful understanding of perturbation theory, a self study will be performed using various text books. “Introduction to Perturbation Methods”, M.H Holmes; Springer; 1995 and “Perturbation Methods”; Ali Nayfeh; John Wiley & Sons ;1973. The second of these books was obtained through the Rensselaer Hartford’s Cole Library. An additional series of material found in the “References” section will be utilized throughout the project.

Computer access with Microsoft Excel and MATLAB will be required to help find and analysis solutions.

Computer access with MAPLE will be required to assist with mathematical analytical analysis.

Computer access with Microsoft Word will be required to produce all project proposals and project write ups.

Computer access with Microsoft Power Point will be required to produce all required presentations.
Proposed Outcome

The proposed outcome of this study is to introduce the perturbation method with the support of simple algebraic equations. Once the perturbation method is introduced, it will be used to develop a set of approximate solutions for the following differential equations variations: Duffing’s equation, the van der Pol equation, a grounded mass connected to a linear and non-linear spring, and finally the chatter in metal turning. The perturbation solution graphs can be compared to analytical or numerical solutions graphs obtained to the same problems throughout the study. This will allow for not only confirmation of the correct application of the perturbation method, but will also allow for the solutions to be compared and contrasted. While finding solutions to these problems is helpful, the most important outcome of the project will be increasing my knowledge of applied mathematics and become more able to solve high degree difficulty problems in the future.
## Deadlines

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<thead>
<tr>
<th>Item No.</th>
<th>Description</th>
<th>Completion Date</th>
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<tbody>
<tr>
<td>1</td>
<td>Functioning Web Portfolio</td>
<td>9/16/2011</td>
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<tr>
<td>2</td>
<td>Introduction to basic Perturbation theory</td>
<td>9/17/2001</td>
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<tr>
<td>4</td>
<td>Research of Algebraic Perturbation problems methods</td>
<td>9/28/2011</td>
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<td>5</td>
<td>Project Proposal Draft</td>
<td>9/30/2011</td>
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<td>6</td>
<td>Solve Duffing Equation</td>
<td>10/10/2011</td>
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<tr>
<td>7</td>
<td>First Progress Report</td>
<td>10/21/2011</td>
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<tr>
<td>8</td>
<td>Solve van der Pol equation</td>
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<tr>
<td>10</td>
<td>Solve non-linear mass spring problem</td>
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<tr>
<td>12</td>
<td>Solve metal chatter problem</td>
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<td>13</td>
<td>Solve by tradition/numerical</td>
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<td>Compare Solutions, analyses differences</td>
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<tr>
<td>16</td>
<td>Final Report</td>
<td>12/16/2011</td>
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References

1) “Introduction to Perturbation Methods”; M.H Holmes; Springer; 1995


3) “Perturbation Methods”; Ali Nayfeh; John Wiley & Sons; 1973

4) “Perturbation Theory & Stability Analysis” University of Twente; T. Weinhart, A singh, A.R. Thornton; May 17, 2010


6) “Some Asymptotic Methods for Strongly Nonlinear Equations”; Ji-Huan He; 2006


8) “Lecture Notes on Nonlinear Vibrations”; Richard Rand; 2005

9) “Applications of Perturbation Methods to Tool Chatter Dynamics”; Ali Nayfeh, Char-Ming Chin, and Jon Pratt; 1998