EFFECT OF GRAVITY ON THE STABILITY OF A ROTATING CANTILEVER BEAM IN A VERTICAL PLANE

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Abstract—The equation of motion of a rotating cantilever beam in a vertical plane is derived based on Hamilton’s principle and assumed mode method. The effect of gravity on the stability of the beam with constant angular velocity is then examined using Bolotin’s method. The regions of instability are determined by converting the resulting equations of boundary frequencies to the standard form of a generalized eigenvalue problem.

1. INTRODUCTION

The vibration of a rotating beam has been studied for a long time in relation to the behavior of rotating blades in turbomachinery or flexible appendages on satellites. Some typical studies, including both rotating uniform beams and pretwisted, tapered beams can be found in the works by Sutherland [1], Renard and Rakowski [2], Likins et al. [3], Anderson [4], Swaminathan and Rao [5], Hodges and Rutkowski [6] and Subrahmanyan et al. [7]. The effect of non-constant rotating speed on the stability behaviors was examined for a radial beam (Kammer and Schlack [8, 9]), using a perturbation technique, and for a pre-twisted, tapered beam (Young [10]) using the method of multiple scales.

In this paper, the equation of motion of a rotating Euler–Bernoulli cantilever beam in a vertical plane is derived, based on Hamilton’s principle and the assumed mode method. The effect of gravity on the stability of rotating cantilever beam in a vertical plane has not been analyzed.

In this paper, the equation of motion of a rotating Euler–Bernoulli cantilever beam in a vertical plane is derived, based on Hamilton’s principle and the assumed mode method. The effect of gravity on the stability of the rotating beam with constant angular velocity is then examined using Bolotin’s method. The regions of instability are determined by converting the resulting equations of boundary frequencies to the standard form of a generalized eigenvalue problem.

2. THEORY AND FORMULATIONS

Figure 1 shows a uniform beam of length $L$ attached to a hub of radius $b$ in a vertical plane. A set of right-handed mutually perpendicular unit vectors, $a_1$, $a_2$, and $a_3$, is assumed to be fixed in the undeformed beam with the $a_1$ vector parallel to the undeformed beam. A second set of mutually perpendicular unit vectors $n_1$, $n_2$, and $n_3$ is assumed to be fixed in a Newtonian frame. The gravitational acceleration $g$ is assumed to be acting vertically downward in the $-n_2$ direction; $n_3$, forming the rotational axis of the hub pointing out of the vertical plane, is parallel to $a_3$ at all times. The position of the hub is measured by an angle $\theta$, defined as the angle subtended by the $a_1$ and $n_1$ unit vectors shown in Fig. 1.

For planar motion in the vertical plane formed by $a_1$ and $n_1$, the position vector of a general point $p$ on the deformed beam is given by

$$p = ra_1 + ua_2.$$  

(1)

For a small deflection, the out of plane motion of the beam in the $a_3$ direction is uncoupled from the motion in the plane formed by the $a_1$ and $n_1$ unit vectors. For simplicity, this out of plane motion will not be considered in the present analysis.

The velocity at the point is

$$v_p = ((b + r)\dot{\theta} + \dot{u})a_2 - u\dot{a}_1,$$  

(2)

where $\theta$ and $\dot{u}$ are defined as

$$\theta = \frac{d\theta}{dt}$$

and

$$\dot{u} = \frac{du}{dt}.$$  

(3)

The kinetic energy $T$ of the beam is

$$T = \frac{1}{2}m \int_0^L \left\{(b + r)\dot{\theta} + \dot{u}\right)^2 + \dot{\theta}^2u^2\right)dr.$$  

(4)
Using the assumed mode method, the quantity $u$ can be expressed as

$$u = \sum_{i=1}^{n} q_i(r)\phi_i(r).$$

where $\phi_i$ are spatial functions that satisfy the geometric boundary conditions at the clamped end of the beam. For the present numerical simulations, the functions are assumed to be the normalized beam functions of a clamped-free beam.

From the assumed functions for $u$, the kinetic energy, $T$, strain energy, $V_s$, and potential energy, $V_a$ and $V_g$, can be expressed in matrix form as

$$T = \frac{1}{2} m \dot{\mathbf{q}}^\top \mathbf{M} \dot{\mathbf{q}} + \frac{1}{2} m \dot{\mathbf{q}}^\top \mathbf{Q} \dot{\mathbf{q}}$$

$$V_s = \frac{1}{2} E I \int_0^L \left( \frac{\partial^2 u}{\partial r^2} \right)^2 \, dr$$

$$V_a = \frac{1}{2} m b \theta^2 (L - r) + \frac{1}{2} m \theta^2 (L^2 - r^2)$$

$$V_g = m g L (b + \frac{1}{2} L) \sin \theta - \frac{1}{2} m g \sin \theta \phi' Q' \dot{\mathbf{q}} + m g \cos \theta Q' \dot{\mathbf{q}}$$

where $\mathbf{M}$, $\mathbf{K}$, $\mathbf{Q}$ and $\mathbf{Y}$ are symmetric matrices defined as

$$\mathbf{(M)}_{ij} = \int_0^L \phi_i \phi_j \, dr$$

$$\mathbf{(K)}_{ij} = \int_0^L \phi_i'' \phi_j'' \, dr$$

$$\mathbf{(Q)}_{ij} = \int_0^L (L - r) \phi_i' \phi_j' \, dr$$

$$\mathbf{(Y)}_{ij} = \int_0^L (L^2 - r^2) \phi_i'' \phi_j'' \, dr$$

The Lagrangian of the beam can then be expressed as

$$L = T - V_s - V_a - V_g.$$
For simplicity, the following dimensionless quantities are introduced:

\[
\tau = t \sqrt{\frac{EI}{mL^4}} \tag{22}
\]

\[
\Omega = \theta \sqrt{\frac{mL^3}{EI}} \tag{23}
\]

\[
\xi = \frac{r}{L} \tag{24}
\]

\[
\bar{b} = \frac{b}{L} \tag{25}
\]

\[
\sigma = \frac{mgL^3}{EI}. \tag{26}
\]

The dimensionless assumed function for a clamped-free beam is

\[
\phi(\xi) = \cosh \lambda_i \xi - \cos \lambda_i \xi + \chi_i (\sinh \lambda_i \xi - \sin \lambda_i \xi), \tag{27}
\]

with

\[
\chi_i = -\frac{\cosh \lambda_i + \cos \lambda_i}{\sinh \lambda_i + \sin \lambda_i}, \tag{28}
\]

and \(\lambda_i\) are the solutions of

\[
1 + \cos \lambda_i \cosh \lambda_i = 0. \tag{29}
\]

The dimensionless \(u\) is given by

\[
\hat{u} = \frac{u}{L} = \sum_{i=1}^{n} \hat{q}_i(\tau) \phi_i(\xi). \tag{30}
\]

The resulting dimensionless equation of motion is

\[
\ddot{\mathbf{q}} + \left( \ddot{\mathbf{M}} + \mathbf{K} + \frac{1}{2} \mathbf{Q}^2 \mathbf{Q} + \frac{1}{4} \mathbf{Q}^2 \mathbf{Q} + \frac{1}{2} \mathbf{Q}^2 \mathbf{Q} - \Omega^2 \mathbf{M} - \sigma \sin \Omega \mathbf{Q} \right) \hat{\mathbf{q}} = \Omega \Psi - \sigma \cos \theta \Phi, \tag{31}
\]

where

\[
(\mathbf{M})_{ij} = \int_0^1 \phi_i(\xi) \phi_j(\xi) d\xi \tag{32}
\]

The homogeneous part of eqn (31) for a rotating beam with constant angular velocity can be re-written in the form of a second order differential equation with periodic coefficients of Mathieu-Hill type:

\[
\ddot{\mathbf{q}} + \mathbf{K} + \mathbf{Q} \left( \frac{1}{2} \mathbf{Q}^2 \mathbf{Q} + \frac{1}{4} \mathbf{Q}^2 \mathbf{Q} + \frac{1}{2} \mathbf{Q}^2 \mathbf{Q} - \Omega^2 \mathbf{M} - \sigma \sin \Omega \mathbf{Q} \right) \hat{\mathbf{q}} = 0. \tag{38}
\]

Using the method presented by Bolotin [13], the region of unstable solutions are separated by periodic solutions having period \(T\) and \(2T\) with \(T = 2\pi/\Omega\). The solutions with period \(2T\) are of greater practical importance. As a first approximation, the periodic solutions with period \(2T\) can be sought in the form [13]

\[
\hat{q} = A \sin \frac{\Omega_\tau}{2} + B \cos \frac{\Omega_\tau}{2}, \tag{39}
\]

where \(A\) and \(B\) are arbitrary vectors.

Substituting eqn (39) into eqn (38) and equating the coefficients of the \(\sin \Omega_\tau/2\) and \(\cos \Omega_\tau/2\) terms, a set of linear homogeneous algebraic equations in terms of \(A\) and \(B\) can be obtained. The condition for non-trivial solutions is

\[
\det \begin{vmatrix} -\frac{1}{2} \Omega^2 \mathbf{M} + \mathbf{K} + \frac{1}{2} \mathbf{Q}^2 \mathbf{Q} + \frac{1}{4} \mathbf{Q}^2 \mathbf{Q} - \Omega^2 \mathbf{M} & -\frac{1}{2} \sigma \mathbf{Q} \\ -\frac{1}{2} \sigma \mathbf{Q} & -\frac{1}{2} \Omega^2 \mathbf{M} + \mathbf{K} + \frac{1}{2} \mathbf{Q}^2 \mathbf{Q} + \frac{1}{4} \mathbf{Q}^2 \mathbf{Q} - \Omega^2 \mathbf{M} \end{vmatrix} = 0. \tag{40}
\]

The above equation can be rearranged in the standard form of a generalized eigenvalue problem

\[
\det \begin{pmatrix} \mathbf{K} & -\frac{1}{2} \mathbf{Q} \\ -\frac{1}{2} \sigma \mathbf{Q} & \mathbf{K} \end{pmatrix} - \Omega^2 \begin{pmatrix} \frac{1}{2} \mathbf{M} - \mathbf{Q} \mathbf{K} - \frac{1}{2} \mathbf{Q} \mathbf{M} & 0 \\ 0 & \frac{1}{2} \mathbf{M} - \mathbf{Q} \mathbf{K} - \frac{1}{2} \mathbf{Q} \mathbf{M} \end{pmatrix} = 0. \tag{41}
\]
The generalized eigenvalues $\Omega^2$ of the above generalized eigenvalue problem can be computed easily by any commercially available eigenvalue package.

4. NUMERICAL RESULTS AND DISCUSSION

The dynamic instability diagrams with period $2T$ for a rotating beam with $\delta$ equal to 0.1, 1 and 10 are presented in Figs 2–4 using a ten-term approximation ($n = 10$) for $u$ which has been found to give numerical values that have converged. A small value of $\delta$ corresponds to a long beam with a small hub. Conversely, a large value of $\delta$ indicates a short beam with a large hub.

The lines in the figures indicate the boundaries between stable and unstable regions. It has been seen from Figs 2–4 that as the value of $\delta$ increases, the slopes of the lines also increase. For the instability diagram of a beam with $\delta = 0.1$, shown in Fig. 2, the behavior of the beam is stable for $\sigma$ less than 8.4484, which is the intercept of the extreme right line with the $\sigma$ axis. The corresponding values of $\sigma$ for $\delta$ equal to 1 (Fig. 3) and 10 (Fig. 4) are 8.4627 and 8.4645. These numerical results show that a rotating beam in
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Fig. 4. The instability diagram for a rotating cantilever beam in a vertical plane with $\beta = 10$.

A vertical plane is not likely to encounter instability of period $2T$ if the beam is stiff (large $EI$), light (small $m$) and short (small $L$) in a normal gravitational field with $g = 9.81$ m/sec$^2$.

5. CONCLUSION

The equation of motion in matrix form of a rotating Euler-Bernoulli cantilever beam in a vertical plane has been derived, based on Hamilton's principle and the assumed mode method. The effect of gravity on the stability of the rotating beam with constant angular velocity is then examined using Bolotin's method. Numerical results show that a rotating beam in a vertical plane is not likely to encounter instability of period $2T$ if the beam is stiff, light and short in a normal gravitational field.

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