Modeling and dynamic analysis of a planetary mechanism with an elastic belt

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Abstract

This paper presents a study of dynamic characteristics of a differential, planetary, path-generating mechanism with a timing belt. By accounting for the belt elasticity and employing Lagrange's equations, a general model of the mechanism is obtained in the form of three nonlinear differential equations. The system’s stiffness is shown to be a periodic time-varying parameter and parametric vibrations are in order. A sample analysis is provided for a particular case, in which the mechanism generates a straight line. Parametric stability of the sample mechanism is investigated based on solving one of the three equations, both in its nonlinear and linearized (Mathieu–Hill’s) forms. Solution is carried out numerically based on expanding the sought-for function into Taylor series on every calculation step. The remaining two equations are then used to find the driving torques. Effects of belt drive speed ratio, belt material damping, and planet link balancing, on the mechanism’s dynamic behavior are investigated.

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1. Introduction

Elasticity of mechanism members leads to undesirable functioning characteristics, such as positioning errors and high levels of noise and vibration. One of the major problems facing mechanism designers is that of ensuring a stable dynamic response of their designs. This requires

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them, in particular, to eliminate the possibility of such an adverse phenomenon, as the parametric resonance. Parametric vibrations are usually excited in a mechanism due to its varying configuration, as well as varying geometric properties during its operation. A system is said to be parametrically stable if its response to the parametric excitation stays bounded for an unlimitedly long time. A typical characteristic of a parametrically excited system is that there are ranges of the excitation amplitude and frequency for which the system vibration amplitudes stay bounded (stable system), and ranges for which they grow unboundedly (parametric resonance). A system can become unstable, undergoing a parametric resonance, for certain relations between the excitation frequency and the system's natural frequency.

In regions of parametric resonance, an exponential growth of vibration amplitudes is usually observed and, obviously, working speeds of mechanisms must lie outside of these critical regions.

The problem of parametric stability normally lends itself to one involving one or more nonlinear differential equations with variable coefficients. An important result of research relevant to the stability of systems is that when a nonlinear system is parametrically excited, the stability of the equilibrium position can usually be decided on the basis of a linearized approximation of its equations of motion [1].

Parametric stability of mechanisms has been the subject of extensive research, especially in the past two decades. For a review of research in this field up to the year 1993, the outstanding reference [1] is recommended. For the general theory of elastodynamics of mechanisms and a large list of references, Chapter 6 of Shabana [2] and the work of Lowen and Chassapis [3] can be used. Also, see [4–10] for sample research work on parametric stability of linkages.

The majority of research in this field deals with theories and techniques for parametric stability analysis and vibration control of linkage and cam mechanisms to account for the effects of the inherent elasticity of their links. The two most widely used techniques for stability analysis are due to Floquet and Liapunov. Zadoks and Midha presented a two-part work [4,5] where they studied parametric stability of an elastic two-degrees-of-freedom machine system with position-dependent inertia and external forcing. The nonlinear equations of motion were linearized about the steady-state response. Floquet theory was used to examine the stability of a slider-crank mechanism with a flexible crankshaft. It was shown that an increase in speed or size of the flywheel leads to an increase in the instability regions. Vulfson [6] employed Liapunov method to investigate parametric stability of mechanical drives with time-varying mass and stiffness properties. The stability conditions obtained were less conservative and had smaller stability margins. Badlani [7] investigated the parametric stability of a flexible slider-crank mechanism with an undamped elastic coupler. It was shown that by accounting for the rotary inertia, the natural frequency and the instability zone are reduced. Inclusion of shear deformation into the formulation was found to widen the instability zone.

Yu and Cleghorn [8] presented a procedure for determining values of critical running speeds that cause a flexible mechanism to become dynamically unstable because of parametric resonance. The finite element method was employed to model each unconstrained flexible link for axial and lateral deformations. Lagrange's equations were used to formulate the global equations of motion. The dynamic stability problem was transformed into a standard eigenvalue problem for which values of all critical running speeds may be computed.

Farhang and Midha [9] presented a model for studying parametric stability of flexible mechanisms. The governing partial differential equation was reduced to the standard Hill's equation by
approximating the coupler deflection using only its first mode of vibration. Floquet theory was applied to the resulting Hill’s equation to investigate parametric stability in a flexible slider-crank mechanism.

A study of effects of drive system characteristics on parametric stability of belt-driven slider-crank mechanisms was performed by Chivate and Farhang [10]. Two models developed. The first model neglects the drive motor speed variations, while the second model accounts for the effects of relatively small speed variations with respect to the average motor operation speed. Floquet theory was used to investigate the effects of different factors (belt stiffness, belt material damping, and belt drive speed ratio) on parametric stability of the driven mechanism.

This work deals with modeling and parametric stability analysis of a planetary, differential, path-generating mechanism with a timing belt drive. Since one of the mechanism members is a belt having a substantially higher flexibility as compared to the remaining links, the latter are assumed undeformable. The rigid-body model of the mechanism has two-degrees-of-freedom. By considering belt elasticity, one more (third) degree of freedom is introduced. Lagrange’s approach is employed to obtain the governing equations of motion. Analysis of the obtained equations showed that the system’s stiffness is a time-varying parameter. Parametric stability of the mechanism is studied on particular mechanism designs, for which the generated path is a straight line. Analysis includes the effects of belt drive speed ratio, belt material damping, and effects of balancing of the floating link (planet) on the mechanism’s dynamic performance and stability.

As far as these authors are aware, the problem of parametric stability of planetary mechanisms with flexible belts has not been addressed in literature.

2. Mechanism kinematics

Planetary gear-linkage mechanisms and, as well, planetary mechanisms, in which the gear pair is replaced by a chain or a belt drive, are used in various branches of machinery. The main feature of these mechanisms is the presence of planets: links that perform complex planar motion, making it possible to obtain a great variety of trajectories of points belonging to these links. These points can directly be used as working (executing) to carry out technological operations or, alternatively, these points can be used to drive other links so that complex displacement functions or paths are generated at the mechanism output [11]. Forms of curves that can be generated by a point of a planet are much more complex for planetary mechanisms with more than one degree of freedom.

The mechanism under study is schematically shown in Fig. 1. Here, pulley 1 and arm $H$ are mounted coaxially on axis $O$. Pulley 2 (planet) is pivoted to arm $H$ at point $A$ and connected to pulley 1 by means of timing belt 3. Lever $AB$ is rigidly attached to pulley 2. End $B$ of lever $AB$ is considered as the working point. Points $S_H$ and $S_2$ locate the centers of mass of arm $H$ and pulley 2 with lever $AB$, respectively.

The rigid-body model of the mechanism has two-degrees-of-freedom. The input links are assumed to be pulley 1 and arm $H$, with the driving moments $M_{M1}$ and $M_{MH}$ applied to them, respectively. Also, the following external actions are applied to the mechanism: reaction moments $M_1$ (at pulley 1), $M_H$ (at arm), $M_2$ (at pulley 2), forces of gravity $m_{Hg}$ of arm and $m_{2g}$ of the body composed of pulley 2 and lever $AB$, and the resisting force $F$ at point $B$ with components $F_X$ and $F_Y$. 
Let the configuration shown in Fig. 1 be the initial position of the mechanism. The relationship between ideal (i.e., without regard to link deformations) angular displacements, velocities, and accelerations of the mechanism links are then found to be:

\[
\begin{align*}
\varphi_2^* &= \varphi_1^* + \varphi_H^* (1 - i), \\
\dot{\varphi}_2^* &= \dot{\varphi}_1^* + \dot{\varphi}_H^* (1 - i), \\
\ddot{\varphi}_2^* &= \ddot{\varphi}_1^* + \ddot{\varphi}_H^* (1 - i),
\end{align*}
\]

where \( i = \frac{R}{r} \).

Parametric equations of trajectory of point \( B \) in the system of coordinates \( XOY \) are:

\[
\begin{align*}
x_B^* &= L \cos \varphi_H + b \cos[\varphi_1 + \varphi_H (1 - i)], \\
y_B^* &= L \sin \varphi_H + b \sin[\varphi_1 + \varphi_H (1 - i)].
\end{align*}
\]  

In the general case, Eq. (2) describe complex curves of epicycloid type, forms of which depend on the geometric and kinematic parameters of the mechanism. In particular, the mechanism can be designed and the input speeds selected so that these epicycloids degenerate into straight lines: a situation frequently used in technology (for example, in presses or in rapier mechanisms of looms). For the trajectory of point \( B \) to be a straight line, coordinate \( y_B^* \) in Eq. (2) must identically be zero. This yields:

\[
L = b \quad \text{and} \quad \varphi_1 = \varphi_H - \frac{2}{i}.
\]  

Obviously, Eq. (3) can be satisfied by any values of the speed ratio \( i \). For the particular case in which \( i = 2 \), pulley 1 becomes stationary.
3. Mechanism dynamic model and equations of motion

Growth of working speeds of modern mechanisms and machines makes it necessary to account for the elastic properties of mechanism links and their influence on the reliability and accuracy of mechanism functioning. First of all in the mechanism under study, one should consider the elastic properties of the timing belt as the most compliant link. Due to tensile deformation of the timing belt, the system gains one more (third) degree of freedom: planet 2 will have an additional (small) angular displacement \( q \). So, actual angular coordinate of planet 2 becomes:

\[
\varphi_2 = \varphi_1 i + \varphi_H (1 - i) + q = \varphi^*_2 + q. \tag{4}
\]

Referred to the axis of planet 2 stiffness \( c \) of the belt drive is:

\[
c = 2c_p r^2 = \frac{2ES}{l}r^2, \quad l = \sqrt{L^2 + (R - r)^2}, \tag{5}
\]

where \( c_p \) is the spring rate of one branch of the belt in tension, \( E = (2.5-9.5) \times 10^8 \text{ N/m}^2 \) and is the elasticity modulus of the belt material, \( S \) is the cross-sectional area of the belt, and \( l \) is the length of a belt branch between pulleys.

In order to set up the governing equations of motion, Lagrange’s approach will be utilized. Lagrange’s equations have the following general form:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} + \frac{\partial G}{\partial q_k} = Q_k, \tag{6}
\]

where \( q_k \) and \( \dot{q}_k \) are generalized coordinates and velocities, respectively; \( Q_k \) is the \( k \)th generalized force, and the subscript \( k \) refers to the associated with it degree of freedom \((k = 1, 2, \ldots, N, \text{where } N \text{ is the number of degrees of freedom})\). In the case under study, \( N = 3 \) and the generalized coordinates are the angular displacements \( \varphi_1, \varphi_H, \text{and } q \). So, \( \varphi_1 \equiv q_1, \varphi_H \equiv q_2, \text{and } q \equiv q_3 \). The terms \( T, D, G, \text{and } Q_k \) of Eq. (6) are given below.

The kinetic energy of the mechanism is

\[
T = \frac{1}{2} \left[ J_1 \dot{\varphi}_1^2 + J_H \dot{\varphi}_H^2 + m_2 (\dot{x}_{S2}^2 + \dot{y}_{S2}^2) + J_2 \dot{\varphi}_2^2 \right]. \tag{7}
\]

Here, \( J_1 \) and \( J_H \) are mass moments of inertia of pulley 1 and arm \( H \) about axis \( O \), \( J_2 \) is the mass moment of inertia of planet 2 with lever \( AB \) about their mass center \( S_2 \).

The \( x \)- and \( y \)-components of velocity of \( S_2 \) in Eq. (7) are:

\[
\dot{x}_{S2} = -(L \dot{\varphi}_H \sin \varphi_H + a \dot{\varphi}_2 \sin \varphi_2), \\
\dot{y}_{S2} = L \dot{\varphi}_H \cos \varphi_H + a \dot{\varphi}_2 \cos \varphi_2. \tag{8}
\]

The Rayleigh dissipative function is

\[
D = \frac{1}{2} \xi \cdot \dot{q}^2, \tag{9}
\]

where \( \xi \) is the coefficient of linear damping.
The potential energy of the mechanism is

\[ G = \frac{1}{2} cq^2 + m_1 g (H_0 + l_1 \sin \phi_H) + m_2 g (H_0 + L \sin \phi_H + a \sin \phi_2). \]  

In Eq. (10), \( H_0 \) defines the reference line, from which gravitational potential energies of links are calculated.

Substituting from Eqs. (4), (7)–(10) into Eq. (6) and carrying out the required mathematical operations yield the governing equations of motion of the mechanism in the following form:

\[
\begin{align*}
J_{11} \ddot{\phi}_1 + J_{1H} \ddot{\phi}_H + J_{1q} \dddot{q} - m_2 i L a \dot{\phi}_H^2 \sin[i(\phi_H - \phi_1) - q] + m_2 i a g \cos \phi_2 &= Q_1, \\
J_{HH} \ddot{\phi}_H + J_{H1} \ddot{\phi}_1 + J_{Hq} \dddot{q} + m_2 L a \dot{\phi}_2 (i \dot{\phi}_1 + \dot{q}) \sin[i(\phi_H - \phi_1) - q] + (m_1 l_1 + m_2 L) g \cos \phi_H \\
&+ m_2 a g (1 - i) \cos \phi_2 = Q_H, \\
J_{qq} \dddot{q} + J_{q1} \ddot{\phi}_1 + J_{qH} \ddot{\phi}_H + \xi \dot{\phi}_1 + cq - m_2 L a \dot{\phi}_H^2 \sin[i(\phi_H - \phi_1) - q] + m_2 a g \cos \phi_2 &= Q_q.
\end{align*}
\]

The following notations were used in the above system of equations:

\[
\begin{align*}
J_{11} &= J_1 + (J_2 + m_2 a^2) i^2, \\
J_{1H} &= \{ (J_2 + m_2 a^2) (1 - i) + m_2 L a \cos[i(\phi_H - \phi_1) - q] \} i, \\
J_{1q} &= (J_2 + m_2 a^2) i, \\
J_{HH} &= J_H + m_2 L^2 + (J_2 + m_2 a^2) (1 - i)^2 + m_2 L a \cos[i(\phi_H - \phi_1) - q], \\
J_{H1} &= (J_2 + m_2 a^2) (1 - i) i + m_2 i L a \cos[i(\phi_H - \phi_1) - q], \\
J_{q1} &= J_2 + m_2 a^2, \\
J_{qH} &= (J_2 + m_2 a^2) (1 - i) + m_2 L a \cos[i(\phi_H - \phi_1) - q].
\end{align*}
\]

Generalized forces \( Q_k \) are now found by equating their work to the work of the external forces and moments on virtual displacements [12]. This yields:

\[
\begin{align*}
Q_1 &= M_{M1} + M_1 + i M_2 - F_x b i \sin \phi_2 + F_y b i \cos \phi_2, \\
Q_H &= M_{MM} + M_H + i M_2 - F_x [L \sin \phi_H + b (1 - i) \sin \phi_2] \\
&+ F_y [L \cos \phi_H + b (1 - i) \cos \phi_2], \\
Q_q &= M_2 - F_x b \sin \phi_2 + F_y b \cos \phi_2.
\end{align*}
\]

Note that, in Eqs. (14)–(16):

\[
\begin{align*}
M_1 &= -|M_1| \cdot \text{sign} \phi_1, \\
M_H &= -|M_H| \cdot \text{sign} \phi_H, \\
\text{and} \quad M_2 &= -|M_2| \cdot \text{sign}(\dot{\phi}_2 - \phi_H).
\end{align*}
\]

The system of differential equations (11)–(13) is nonlinear, and the methodology of its solution depends on the particular problem of investigation. If, for instance, functions \( \phi_1 \) and \( \phi_H \) are not given, it is required that characteristics of the driving torques \( M_{M1} \) and \( M_{MM} \) be introduced. This yields a new system of equations, the order of which is determined by the form of driving-torque characteristics used. If, otherwise, \( \phi_1 \) and \( \phi_H \) are known, it is only required to solve Eq. (13), while Eqs. (11) and (12) are then used to find the driving torques \( M_{M1} \) and \( M_{MM} \). The latter problem statement is more frequently solved in real designing of similar mechanisms since it simplifies the
general solution and allows for qualitative evaluation of the dynamic characteristics of the system. In this connection, below is assumed that the angular velocities $\dot{\phi}_1$, $\dot{\phi}_H$ are known and, moreover, are uniform. Furthermore, in what follows only the case of straight-line motion of point $B$ will be analyzed, with $F_Y$ set to zero and $F_X$ directed in the negative $X$-direction and acting on $B$ only during the left-to-right stroke of point $B$ (see Fig. 1).

Under the above assumptions and with conditions in Eq. (3) fulfilled, Eq. (13) becomes:

$$J_{qq} \ddot{q} + \ddot{\phi}_H^2 \sin(2\phi_H - q) + m_L a \cos(\phi_H - q) = M_2 + F_X L \sin(\phi_H - q).$$

In real mechanisms of the type under study, values of $q$ are small and the nonlinear equation (17) can be linearized based on the following approximations:

$$\sin(2\phi_H - q) \approx \sin 2\phi_H - q \cos 2\phi_H,$$
$$\cos(\phi_H - q) \approx \cos \phi_H + q \sin \phi_H,$$
$$\sin(\phi_H - q) \approx \sin \phi_H - q \cos \phi_H.$$ 

Incorporating the above approximations into Eq. (17) results in a linear differential equation with variable coefficients (Mathieu–Hill type):

$$J_{qq} \ddot{q} + \dot{\phi}_H^2 \sin 2\phi_H + m_L a \cos \phi_H + F_X L \sin \phi_H)q = M_2 + m_L a \phi_H^2 \sin 2\phi_H - m_L a \cos \phi_H + F_X L \sin \phi_H.$$ 

The character of solution of equations similar to Eq. (19) is well-investigated and for applied mechanical problems the most important result consists in the fact that the system stiffness (and, consequently, its natural frequency) is a periodic function. In this case the system vibrations are parametric; they depend on the amplitude of deviation of system’s stiffness from its mean value. It then becomes possible for a succession of regions of parametric resonance to appear in the vicinities of the stiffness variation frequencies $\omega$ (the excitation frequencies). Theses frequencies are defined from:

$$\omega = \frac{2k_0}{j}, \quad (j = 1, 2, 3, \ldots), \quad k_0 = \sqrt{\frac{c}{J_{qq}}},$$

where $k_0$ is the mean natural frequency.

Two values of the excitation frequency $\omega$, equal to $\dot{\phi}_H$ and $2\dot{\phi}_H$, appear explicitly in Eq. (19).

It is worth reminding that Eqs. (17) and (19) are equivalent to each other only if $q$ is small and, in the general case, analysis of the mechanism must be based on Eq. (17).

### 4. Solution procedure outline and main results

In solving each of Eqs. (17) and (19) a numerical method was used which is based on expanding the sought-for solutions into Taylor series on every calculation step. The solution procedure can be illustrated on Eq. (17) as follows. Let $q(t)$ and $\ddot{q}(t)$ be known at an arbitrary time instant $t$. Using Eq. (17) we find the second and third time-derivatives of $q(t)$:

$$\ddot{q}(t) = \frac{1}{J_{qq}} [-\dot{\phi}_H^2 \sin(2\phi_H - q) + m_L a \cos(\phi_H - q) + M_2 + F_X L \sin(\phi_H - q)],$$

(21)
\[
\ddot{q}(t) = \frac{1}{J_{qq}} \left[ -\xi \cdot \ddot{q} - cq + m_2 La \ddot{\phi}_H^2 (2\phi_H - \ddot{q}) \cos(2\phi_H - q) + m_2 a g (\ddot{\phi}_H - \ddot{q}) \sin(\phi_H - q) \\
+ F_x L (\ddot{\phi}_H - \ddot{q}) \cos(\phi_H - q) \right].
\] (22)

Proceeding further to find more higher-order derivatives and using Taylor series determine \(q(t + \Delta t)\) and \(\dot{q}(t + \Delta t)\) at the end of the calculation step:

\[
q(t + \Delta t) = q(t) + \dot{q}(t) \Delta t + \frac{1}{1 \cdot 2} \ddot{q}(t) \Delta t^2 + \frac{1}{1 \cdot 2 \cdot 3} \dddot{q}(t) \Delta t^3 + \cdots
\] (23)

\[
\dot{q}(t + \Delta t) = \dot{q}(t) + \dddot{q}(t) \Delta t + \frac{1}{1 \cdot 2} \dddot{q}(t) \Delta t^2 + \cdots
\] (24)

In this manner, a calculation cycle can be organized using Eqs. (21)–(24) over any given time interval \(t_0\).

This method is more laborious as compared to other methods since it requires finding derivatives of higher order in analytical form. At this expense, however, this method allows for a higher accuracy of calculations, provided that a sufficiently large number of the Taylor series terms are included [13]. It is usually sufficient to find derivatives of up to the fourth or fifth order.

Calculation procedures were programmed in Borland C++ language with the following data as program inputs: geometric dimensions of mechanism links, physical properties of materials used, damping coefficient, and working speeds of the mechanism.

In analyzing the dynamic behavior of the planetary mechanism, the question on the first place was the issue of possible occurrence of parametric resonances in the vicinity of the mechanism working speed \(\dot{\phi}_H\). Calculations were carried out for a number of sample mechanism alternatives to clarify this issue.

Below are representative analysis results obtained for particular mechanism designs having size dimensions of \(L = b = 0.15\) m (see Fig. 1). The belt used had an elasticity modulus of \(E = 3.10^8\) N/m\(^2\) and cross-sectional dimensions of \(3.0 \times 20\) mm. Also, the resisting force was assumed to be \(F_x = 30\) N when point \(B\) moves from left to right and \(F_x = 0\) otherwise. These parameters are not altered hereafter.

### 4.1. Undamped system

Table 1 cites maximum amplitudes \(q_{\text{max}}\) and \(q_{0\text{max}}\) obtained for various working speeds \(\dot{\phi}_H\) of the sample mechanism with the assumption that damping is absent (\(\xi = 0\)). Values \(q_{\text{max}}\) are

<table>
<thead>
<tr>
<th>(j)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>8</th>
<th>9</th>
<th>11</th>
<th>12</th>
<th>16</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\dot{\phi}_H), (\text{s}^{-1})</td>
<td>369.0</td>
<td>246.0</td>
<td>184.5</td>
<td>147.6</td>
<td>92.2</td>
<td>82.0</td>
<td>67.1</td>
<td>61.5</td>
<td>46.1</td>
<td>30.7</td>
</tr>
<tr>
<td>(q_{\text{max}}), rad</td>
<td>5.0</td>
<td>1.25</td>
<td>3.90</td>
<td>0.70</td>
<td>0.25</td>
<td>0.08</td>
<td>0.05</td>
<td>0.05</td>
<td>0.03</td>
<td>0.018</td>
</tr>
<tr>
<td>(q_{0\text{max}}), rad</td>
<td>10000</td>
<td>1.25</td>
<td>12.50</td>
<td>0.70</td>
<td>0.25</td>
<td>0.08</td>
<td>0.05</td>
<td>0.05</td>
<td>0.03</td>
<td>0.018</td>
</tr>
</tbody>
</table>
maximum amplitudes computed using the nonlinear equation (17), whereas \( q_{0 \text{max}} \) are maximum amplitudes found by using the linearized equation (19). This mechanism was synthesized so that \( R = r = 0.06 \text{ m} \) \( (i = 1) \). Other mechanism parameters, such as dimensions, masses and mass–moments of inertia of pulleys, arm \( H \), and lever \( AB \), were selected tentatively and resulted in a mean natural frequency of \( k_0 = 369 \text{ rad/s} \).

Working speeds \( \dot{\phi}_H \) were taken in the centers of regions of parametric resonance, that is, on the basis of Eq. (20), \( \dot{\phi}_H = 2k_0/j \). Calculations for each \( j \) were carried out over the first seven cycles of motion.

Analysis of the calculation results showed that, in the region with \( j = 2 \) and a frequency of stiffness variation of \( 2\dot{\phi}_H \), i.e., \( \omega = 2k_0 \), a principal parametric resonance occurs and amplitudes reach prohibitively large values (see Table 1). Characteristically in this case, there is a sound difference between solutions of the nonlinear and the linearized equations of motion: solution of the nonlinear equation is bounded, whereas that of the linearized equation continues to increase unlimitedly. An analogous situation is observed for \( j = 4 \). Increasing the calculations duration to forty cycles allowed to ascertain the fact that, for all regions with even values \( j > 4 \), vibration amplitudes, though slowly increasing, are unbounded, whereas for any region with an odd value of \( j \), vibration amplitudes remain constant.

### 4.2. Effect of damping

It is important to note the role of damping in decreasing parametric vibrations and in affecting the way they develop in, both in the nonlinear and the linearized models. Fig. 2 shows vibration phase diagrams for the region \( j = 2 \), with \( \xi = 0 \) (curves 1 and 2) and with \( \xi = 0.5 \) (curves 3 and 4). Curves 2 and 4 were obtained using the nonlinear equation (17), while curves 1 and 3—using the
linearized equation (19). These curves demonstrate the fact that it is possible, through damping, to substantially suppress resonant parametric vibrations. This is especially true for resonance regions with \( j > 10 \), near which real working speeds \( \dot{\phi}_H \) of mechanisms under study usually lie.

### 4.3. Effect of speed ratio

One of the main issues in designing of planetary mechanisms is the task of selecting a suitable speed ratio \( i = R/r \) based on evaluating its effect on the dynamic characteristics of the designed mechanism. This issue has been dealt with in this study. For the analysis here, all other sample-mechanism parameters were kept unchanged (see Section 4), and the speed ratio \( i \) was varied. This, of course, required that the driving pulley’s speed \( \dot{\phi}_1 \) be recalculated with each new value \( i \) using Eq. (3) so that the desired straight-line path of point \( B \) is maintained. The arm speed was fixed at 300 rpm, i.e., \( \dot{\phi}_H = 31.416 \) rad/s.

Table 2 shows results of calculations for different speed ratios \( i: \dot{\phi}_1, k_0, \) maximum and minimum driving torques \( M_{M1} \) and \( M_{MH} \), maximum and minimum amplitudes of \( q \), and maximum deviation

<table>
<thead>
<tr>
<th>No.</th>
<th>( i )</th>
<th>( \dot{\phi}_1, ) s(^{-1} )</th>
<th>( k_0, ) s(^{-1} )</th>
<th>( M_{M1}, ) Nm</th>
<th>( M_{MH}, ) Nm</th>
<th>( q, ) rad</th>
<th>( y_B, ) m</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>15.7</td>
<td>198</td>
<td>+22</td>
<td>+29</td>
<td>+0.085</td>
<td>+0.006</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>−38</td>
<td>−5</td>
<td>−0.044</td>
<td>−0.008</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>10.7</td>
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tions \( y_B \) of trajectory of point \( B \) from the \( X \)-axis. Note that \( y_B \) is the deviation of point \( B \) from its desired path and it defines the path-generation error induced by vibrations.

Table 2 can be used to select a suitable mechanism design from the viewpoint of its dynamic behavior to provide a minimum level of loads in mechanism links and a higher accuracy of path generation. If, for instance, minimization of the driving torques is the design criterion, then alternative No. 4, for which \( i = 1 \), should be selected (Ignore alternative No. 8 for now). Alternative No. 4 would, as well, have quite acceptable values of other dynamic characteristics.

4.4. Effect of balancing of planet

Inspection of Eqs. (17) and (19) shows that their structure depends substantially on the value of the parameter \( a \), which locates the center of mass \( S_2 \) of the body composed of pulley 2 and lever \( AB \) (see Fig. 1). It is always possible to design this body (planet) so that \( S_2 \) coincides with axis \( A \), hence balancing it on its axis of rotation relative to arm \( H \) and setting \( a \) to zero. By doing so, planet’s mass and mass–moment of inertia would increase. However, the terms that are multiplied by \( a \) in Eqs. (17) and (19) disappear and the level of dynamic loads in the mechanism links would decrease. Alternative No. 8 in Table 2 is the only one with \( a = 0 \). By comparing this alternative with alternative No. 4, it is seen that driving-torque amplitudes are substantially reduced.

To give more insight into the effect of balancing, Figs. 3–5 are given that show graphs of \( q \) and driving torques \( M_{M1} \) and \( M_{MH} \) versus time for mechanisms with an unbalanced planet \((a \neq 0, \text{ curves } 1)\) and with a balanced planet \((a = 0, \text{ curves } 2)\). It is seen from these figures that, in all cases, balancing of the planet has highly desirable effects on the dynamic characteristics of the planetary mechanism.
5. Conclusion

This work presented a dynamic model and an analysis procedure for a planetary mechanism with a timing belt drive. The model accounted for three degrees of freedom of the mechanism, including the one resulting from the tensile elasticity of the belt. The model was obtained by
employing Lagrange's principle. The system stiffness was shown to be a position-dependent periodic function, and parametric vibrations are in order. A study of parametric stability of a sample mechanism was carried out both on the original (nonlinear) and a linearized model. By analyzing an undamped sample model, and with the relationship $\omega = 2k_0/j$ in mind to define possible resonance regions ($\omega$ is the excitation frequency and $k_0$ is the mean natural frequency, $j = 1, 2, 3, \ldots$), it was found that the system is unstable in all regions with even values of $j$, with a principal resonance occurring in the region $j = 2$. The system was shown to be stable in all regions with odd values of $j$. It was further shown that, by introducing a damping factor of 0.5, parametric resonances are effectively suppressed; this is especially true for resonance regions with $j > 10$. The belt drive speed ratio was shown to highly affect the driving-torques amplitudes. The positive effect of balancing of the planet link on the mechanism dynamic behavior was demonstrated.

The presented dynamic analysis of the planetary mechanism with a straight-line motion of the working point contains all stages of calculation, and the used methodology of modeling and analysis can be applied to more general problems.

References