Conduction Heat Transfer from a Hot Tub

by

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Semester Project for Course MANE-7000 AEM2, Spring, 2006

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ABSTRACT

Maximum Steady State Temperature, heat Transfer at a typical operating temperature, time to heat and time to cool are calculated for a hot tub. The water is treated as a lumped capacitance, and heat transfer out of the tub is approximated as one dimensional heat transfer. The temperature profile across the wall as the hot tub is heated is solved using Duhamel's method and the explicit finite difference scheme. Linearity of the temperature profile is investigated, and an assumed continuously linear temperature profile is used in calculating the time for the water to cool.

1. Introduction

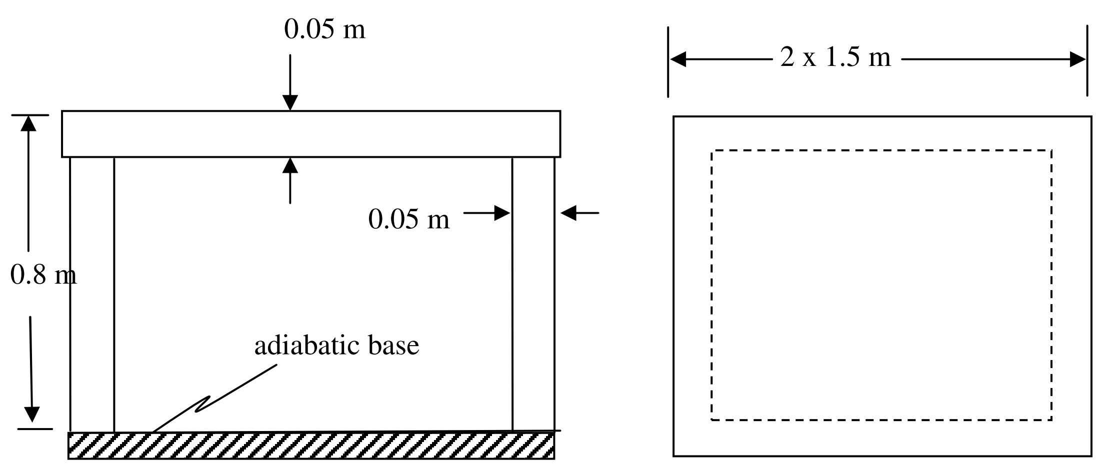
The goal of this paper is to determine the maximum temperature (relative to an ambient air temperature) that can be maintained by a hot tub, determine heat loss at a typical temperature, determine the amount of time to heat the hot tub based on given conditions, and cooling time from a steady state condition.

In all cases, the problem will be bounded using simplified analysis and assumptions, and then simplifying assumptions will be removed to achieve more accurate analytical and numerical solutions. Numerical solutions will be validated by similar or simplified analytical solutions. In all cases many simplifying assumptions will remain. This paper aims to focus on the losses due to conduction, so convective effects will not be considered.

The dimensions and geometry of the hot tub are simplified to a box with walls / cover of constant thickness. The water in the hot tub is treated as a lumped thermal capacitance. The heat generated is based on the electrical requirements of the heater, assuming 100% efficient conversion from electricity to heat. For all cases, the base of the hot tub is assumed to be adiabatic. The outer wall surfaces are assumed to be at the ambient outside temperature.

2. History

I moved into a new house this past October that has a hot tub in the back yard. So it occurred to me to look into heat loss from the tub and time to heat.



Cross section side view (not to scale) **Top view** (not to scale)

Figure 1: Dimensions and Geometry of Hot Tub Considered

3. Maximum Steady State Water Temperature – Cover On, Analytical

3.1 Maximum Steady State Water Temperature – Theory

The problem is set up as a one dimensional steady state heat transfer problem (transfer through the cover and the walls, with the one-dimensional heat fluxes then being applied to the applicable surface area (the surface area of the cover and the walls)

The one dimensional steady state heat equation simplifies to:

$$\frac{\partial}{\partial x}\left(k \cdot \frac{\partial T}{\partial x}\right) = 0 \tag{1}$$

The solution is:

$$q'' = \frac{k(T_0 - T_1)}{x} \tag{2}$$

Applied to this problem, Heat generated equals heat transfer out of the tub at steady state:

$$\dot{Q}_{generated} = q'' \cdot (A) = \frac{k \cdot A \cdot \Delta T}{x} \tag{3}$$

Solving for Temperature:

$$\Delta T = \frac{\dot{Q}}{\left(\frac{k \cdot A}{x}\right)} \tag{4}$$

3.2 Steady State Temperature – Results

The rated power for the hot tub is 220 Volts at 60 Amps, so the total power output is 13,200 Watts. All power is assumed to be converted to heat and transferred to the water. The properties of the actual hot tub insulating materials are not known, but they are assumed to be similar to Rigid urethane foam (two-part mixture), the properties of which are listed in Appendix A of Reference [1]. The property of interest here is thermal conductivity, k = 0.026 W/m K. The wall thickness as stated above is 0.05 m and the wall surface area is 6.16 m^2. The resulting steady state temperature is 4120ºC above the outside temperature

3.3 Steady State Temperature Discussion

The result of 4120ºC is surprisingly high. An approximate worst case differential in winter is about 60ºC. It does make sense in that the tub must maintain temperature with the cover off, when the top surface of the water is directly exposed to the outside air, and suggests that the tub is well insulated.

A typical temperature change, if the hot tub is maintained at a low setting to save energy, is 10ºC to 20ºC. Since the magnitude of the temperature change is small compared to the steady state temperature and the total temperature is also very small relative to the steady state temperature, it is possible that the Temperature increase of the tub will be linear with time. This will be investigated in the first part of the next section.

4. Heating the Hot Tub

The wall temperature will be calculated both analytically and numerically and the results will be compared to validate that both answers make sense and to validate the use of the numerical method for more complex cases. Two cases will be considered. For case 1, the wall and the water are initially at the outside air temperature. For case 2, the water is initially 20º C higher than the outside air temperature and there is a steady state temperature distribution through the wall. For both cases, the water temepratuure is increased by 20ºC

4.1 Upper and Lower Bounds

4.1.1 Theory

4.1.1.1 Lower Bound of time to heat – Perfectly insulated hot tub

The lowest amount of time to heat the water in the hot tub is obtained assuming that all heat generated goes into the water and is not transferred out through the walls. The equation for heating the water with no losses is:

$$\dot{Q} = m \cdot c_{p,water} \cdot \frac{dT}{dt} \tag{5}$$

Where $c_{p,water}$ is the specific heat capacity of water, and m is the total water mass. Solving for t:

$$t = \frac{m \cdot c_{p,water} \cdot \Delta T}{\dot{Q}_{heater}} = \frac{\Delta E_{water}}{\dot{Q}_{heater}} \tag{6}$$

4.1.1.2 Upper Bound – Maximum Heat Loss:

The upper bound of time to heat the hot tub adds the heat required to heat the water, the heat required to establish a steady state temperature distribution across the wall at the final temperature (Equation (7)), and subtracts the heat loss at the final temperature (Equation (8)) from the heater output.

$$\Delta E_{wall} = \rho_{wall} \cdot V_{wall} \cdot c_{p,wall} \cdot \frac{\Delta T_{water}}{2} \quad (7)$$

$$\dot{Q}_{\max\, walls} = \frac{k_{wall} \cdot A \cdot (T_{water} - T_{outside})}{L} \quad (8)$$

$$t = \frac{\Delta E_{wall} + \Delta E_{water}}{\dot{Q}_{heater} - \dot{Q}_{\max\, walls}} \quad (9)$$

This is a conservative estimate because the heat transfer from the hot tub will vary with the water temperature, and is assumed to be at the maximum value for the total time to heat.

4.1.2 Upper and Lower Bound Results:

The case considered was to increase the hot tub water temperature by 20ºC. For the perfectly insulated case, the time to heat is 9323 seconds (2.59 hours).

For the maximum heat loss case, a number of temperatures were considered to determine the effects of heat loss on heating time. The results are summarized in Table 1 below:

Table 1: Time to Raise Water Temperature by 20º C for Various Conditions

Time to Raise Water Temp 20 ºC							
t no heat loss (sec)	t max heat loss (sec)	Δ T init (ºC)	$T_{increase}$ (ºC)	max heat loss (W)	% difference	halfway point	Max Δ T (ºC)
9323.364	9385.985	0	20	64.064	0.6716637	9354.674	20
9323.364	9397.443	5	20	80.08	0.7945575	9360.403	25
9323.364	9408.929	10	20	96.096	0.9177518	9366.146	30
9323.364	9420.443	15	20	112.112	1.0412475	9371.903	35
9323.364	9431.985	20	20	128.128	1.1650459	9377.674	40
9323.364	9443.556	25	20	144.144	1.289148	9383.46	45
9323.364	9455.155	30	20	160.16	1.413555	9389.259	50
9323.364	9466.782	35	20	176.176	1.538268	9395.073	55
9323.364	9478.438	40	20	192.192	1.663288	9400.901	60

4.1.3 Discussion of Results

As shown above, the percent difference in time to heat for maximum heat loss vs no heat loss is very small, ranging from 0.7% to 1.7%. Since the temperature change with no heat loss is linear, and the time to heat with maximum heat loss is very close, the temperature increase in the water can be assumed to be linear. In the following sections, the water temperature (equal to the temperature at the inner wall surface) will be increased linearly with time based on the average of the no heat loss and maximum heat loss times to heat.

4.2 Temperature Distribution in Hot Tub Wall When Increasing Temperature - Analytical

4.2.1 Wall Temperature – Theory

The solutions must satisfy the heat equation. The problem can be nondimensionalized for simplicity with dimensionless Temperature, time and distance as follows [1]:

$$u^* = \frac{T - T_o}{T_i - T_o} \tag{10}$$

Where T is the temperature at any location and time (x,t), T_o is the temperature at the outside wall of the tub and $T_i - T_0$ is the maximum temperature difference.

$$x^* = \frac{x}{L} \tag{11}$$

Where x is position within the wall and L is the length across the wall thickness.

$$t^* = \frac{\alpha \cdot t}{L^2} \tag{12}$$

Where α is the thermal diffusivity of the wall. and t is time.

The non-dimensionalized governing equation becomes

$$\frac{\partial^2 u^*}{\partial x^{*2}} = \frac{\partial \theta^*}{\partial t^*} \tag{13}$$

4.2.1.1 Case 1 – Water initially at outside air temperature

The boundary conditions are:

$$\begin{aligned} u^*(1,t) &= 0 \\ u^*(0,t) &= \frac{d\theta^*}{dt^*}t \\ u^*(x,0) &= 0 \end{aligned} \tag{14}$$

Since the boundary condition at the inner surface is a continuous function of t, Duhamel's Method of Superposition is used to solve the problem. The solution to the problem with a simpler boundary condition $\theta^*(0,t) = 1$ is obtained. This solution is worked out in reference [2] , as follows:

Straight separation of variables does not give a good result, so the problem is transformed. U(x) is expressed as the sum of two expressions, the steady state solution and another expression of x* and t*

$$U(x^*,t^*) = \tilde{U}(x^*) + v(x^*,t^*) \tag{15}$$

Where Û(x*) is the steady state solution, which works out to:

$$\tilde{U}(x^*) = 1 - x^* \tag{16}$$

and v(x*,t*) is the other expression that can be solved. This can be rewritten as:

$$v(x^*,t^*) = U(x^*,t^*) - \tilde{U}(x^*) \tag{17}$$

The boundary and initial conditions are transformed by the second equation and become:

$$v(1,t^*) = 0 - 0 = 0 \tag{18}$$

$$v(0,t^*) = 1 - 1 = 0 \tag{19}$$

$$v(x^*,0) = 0 - (1 - x^*) = x^* - 1 \tag{20}$$

v(x*,t*) is then solved by seperation of variables. It is assumed to be a product of two functions, X(x*) and T(t*). The product is plugged back into the heat equation and both sides are divided by X x Tyielding:

$$\frac{1}{X}\cdot\frac{d^2X}{dx^2} = \frac{1}{T}\cdot\frac{dT}{dt} = -k^2 \tag{21}$$

Where $-k^2$ is a constant, because for the two functions to be equal for all values of x* and t*, the two equations must be equal to a constant. The – sign is chosen because typically exponentials found are of the decaying variety. Equation 21 can then be written as equations (22) and (23):

$$\frac{d^2X}{dx^{*2}} + k^2X = 0 \tag{22}$$

$$\frac{dT}{dt^*} + Tk^2 = 0 \tag{23}$$

The solutions of which are

$$X = A\cdot\sin kx^* + B\cdot\cos kx^* \tag{24}$$

$$T = C\cdot\exp(-k^2t^*) \tag{25}$$

From the boundary conditions:

$$v(0,t^*) = 0 \rightarrow B = 0 \tag{26}$$

$$v(1,t^*) = 0 \rightarrow k = n\cdot\pi \tag{27}$$

v(x*,t*) should then be expressed as a linear combination of X and Y

$$v(x^*,t^*) = \sum_{n=1}^{\infty} a_n \cdot X_n \cdot T_n \tag{28}$$

$$X_n = \sin n\pi x^* \tag{29}$$

$$T_n = \exp(-n^2\pi^2 t^*) \tag{30}$$

Satisfying the initial condition:

$$v(x^*,0) = x^* - 1 = \sum_{n=1}^{\infty} a_n \cdot \sin n\pi x^* \cdot \exp(-n^2\pi^2 \cdot 0) \tag{31}$$

This is a Fourier sin series, so a_n is solved as:

$$a_n = \frac{\int_0^1 (x^* - 1) \cdot \sin n\pi x^* \cdot dx^*}{\int_0^1 [\sin n\pi x^*]^2 \cdot dx^*} = -\frac{2}{n \cdot \pi} \tag{32}$$

So the solution to v(x*,t*) is:

$$v(x^*,t^*) = -\sum_{n=1}^{\infty} \frac{2}{n \cdot \pi} \cdot \sin n\pi x^* \cdot \exp(-n^2\pi^2 t^*) \tag{33}$$

Plugging back into equation (15) (adding initial condition), U becomes:

$$U(x^*,t^*) = 1 - x - \frac{2}{\pi} \cdot \sum_{n=1}^{\infty} \frac{1}{n} \cdot \sin(n\pi x) \cdot \exp(-n^2\pi^2 t) \tag{34}$$

The Duhamel Superposition Method, for a boundary condition that is a function of t with no discontinuities provides the following solution for u(x*,t*) as a function of U(x*,t*-τ) (for all t* in U, replace with t*-τ) as follows:

$$u(x^*,t^*) = \int_0^{\theta} U(x, t - \tau) \cdot F'(\tau) \cdot d\tau \tag{35}$$

Substituting in U(x*,t*-τ):

$$u(x^*,t^*) = \int_0^{t^*} \left(1 - x^* - \frac{2}{\pi} \cdot \sum_{n=1}^{\infty} \frac{1}{n} \cdot \sin(n\pi x^*) \cdot \exp(-n^2\pi^2 (t^* - \tau) \right) \cdot F'(\tau) \cdot d\tau \tag{36}$$

Taking out all non-functions of τ,

$$u(x^*,t^*) = (1 - x^*) \cdot F'(\tau) \cdot \int_0^z d\tau + $$
$$- \frac{2}{\pi} \cdot \sum_{n=1}^{\infty} \frac{1}{n} \cdot \sin(n\pi x^*) \cdot F'(\tau) \cdot \int_0^z \exp(-n^2\pi^2(t^* - \tau) \cdot d\tau \tag{37}$$

(Note that F'(τ) is a constant as the temperature is assumed to rise linearly). Note that the upper limit of integration has changed. The limit of integration z is defined as:

$$z = \begin{cases} t^*, t^* < \tau_{ss} \\ \tau_{ss}, t^* > \tau_{ss} \end{cases} \tag{38}$$

τ_{ss} is the time at which the boundary temperature reaches steady state (the tub is heated to and maintained at a desired temperature). The temperature becomes constant at time τ_{ss}. There is a term associated with time after τ_{ss}:

$$\int_{\tau_{ss}}^{t^*} \left(1 - x^* - \frac{2}{\pi} \cdot \sum a_n \cdot \sin(n\pi x^*) \cdot \exp(-n^2\pi^2(t^* - \tau) \right) \cdot F'(\tau) \cdot d\tau = 0 \tag{39}$$

This term is equal to zero because the change in the value of the boundary condition F'(τ) is zero in this range.

The final closed form of the solution is:

$$u(x^*,t^*) = (1 - x^*) \cdot F'(\tau) \cdot z$$
$$- \frac{2 \cdot F'(\tau)}{\pi} \cdot \sum_{n=1}^{\infty} \frac{1}{n} \cdot \sin(n\pi x^*) \cdot \exp(-n^2\pi^2 t^*) \cdot \frac{(\exp(n^2\pi^2 z) - 1)}{n^2\pi^2} \tag{40}$$

4.2.1.2 Case 2 – Water initially 20ºC above outside air temperature, initial steady state heat flux through walls

The boundary conditions are:

$$u^*(1,t^*) = 0$$
$$u^*(0,t^*) = \begin{cases} \frac{du^*}{dt^*} t, t^* < \tau_{ss} \\ 1, t^* > \tau_{ss} \end{cases} \tag{41}$$
$$u^*(x^*,0) = 0.5 - 0.5x^*$$

The solution is similar to that for Case 1, except the initial condition is added to the solution and F'(τ) in this case is half of what it was in the previous case, because the relation of absolute to dimensionless temperature has changed. The solution for this case is:

$$u(x^*,t^*) = (1-x^*)\cdot F'(\tau)\cdot z + 0.5 - 0.5x^*$$
$$-\frac{2\cdot F'(\tau)}{\pi}\cdot\sum_{n=1}^{\infty}\frac{1}{n}\cdot\sin(n\pi x^*)\cdot\exp(-n^2\pi^2 t^*)\cdot\frac{(\exp(n^2\pi^2 z)-1)}{n^2\pi^2} \qquad (42)$$

4.2.2 Analytical Solution

4.2.2.1 Case 1 Solution:

The first case is to raise the temperature of the water by 20ºC starting at the outside temperature. The normalized dimensionless temperature (u*) will be 0 with the water at air temperature and 1 when it reaches 20ºC above the outside air. The wall thickness is 0.05 meters, so x*=0 is the inside wall of the tub and x*=1 is the outside surface of the tub. The conversion from t* to actual time is $t^* = 1.4217\text{x}10^{-4}$ t. For t < τ_{ss}, F'(τ)=0.7523 and τ_{ss} = 1.3293 (based on 9350 sec heating time).

The solution of the above function u(x*,t*) defined in the previous section is checked by verifying that the boundary conditions are satisfied. Figure 2 shows a 3-d plot of the solution.

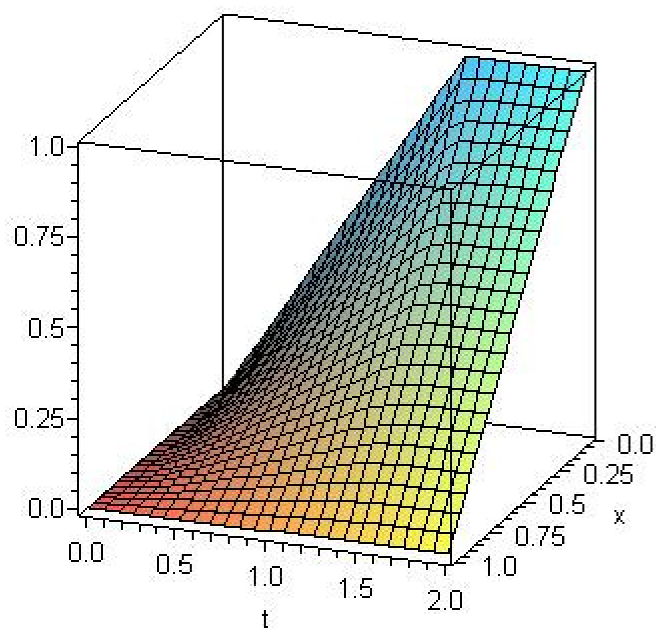


Figure 2: Three Dimensional Plot of u*(x*,t*) for Case 1

From Figure 2, it appears that the u(1,t*) = 0 and u(x*,0) = 0 are met. The maximum value of u appears to be right around 1. A two dimensional plot of u* vs t* at x = 0 is shown below:

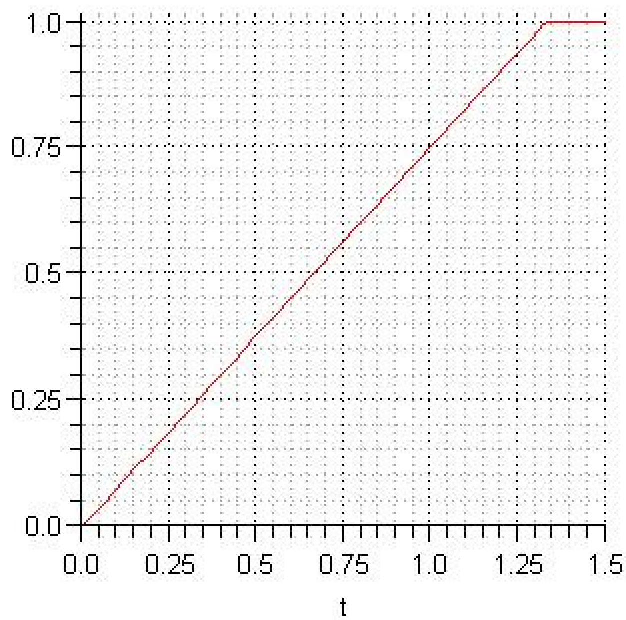


Figure 3: Plot of Analytical Solution of u*(0,t*) for Case 1

Figure 3 shows that the upper limit of the dimensionless temperature is 1, as expected and that the temperature ramps up linearly with a slope of about 0.75 initially and then reaches steady state at t* near 1.3.

From Figure 2, the temperature distribution across x starts out with a parabolic look to it and eventually becomes a linear temperature distribution across the wall. Figure 4 shows u* vs x* for values of t* from 0.2 to 1.6 (steps of 0.2):

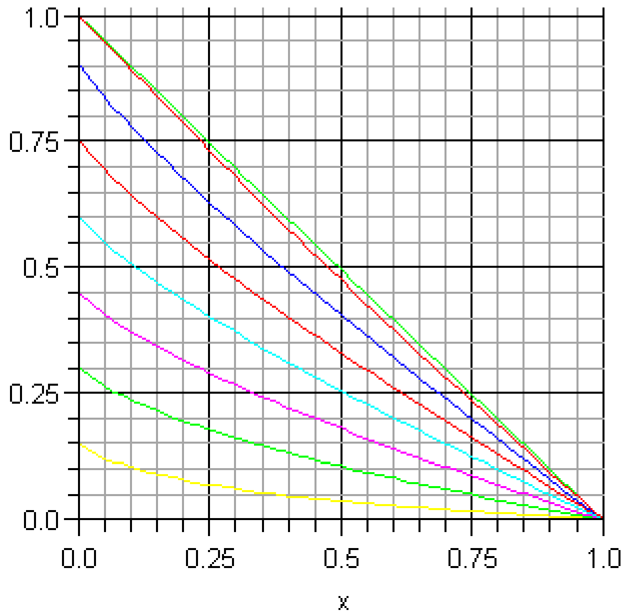


Figure 4: 2-D Plot of Analytical Temperature Profile for Case 1.

To determine how quickly the temperature distribution becomes linear, the ratio of u* at x = 0.5 divided by u* at 0 is plotted over time. As the temperature distribution becomes linear, the ratio will approach 0.5.

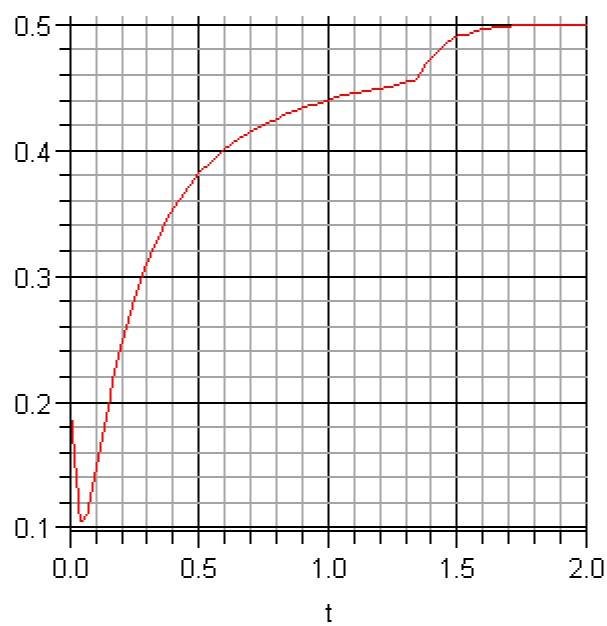


Figure 5: u*(0.5,t*)/u*(0,t*) vs t* for Analytical Solution Case 1

As shown, the temperature distribution is initially not very linear and the linearity increases rapidly at first and then appears to be leveling off or approaching 0.5 more slowly. The likely cause of the tailoff is that the rate of increase in temperature at the surface becomes smaller with time relative to the total temperature magnitude. The initial section of the curve is likely approximation error as the series solution goes from 1 to 0 at the first instant in time (The ratio is initially 0/0 = 1).

4.2.2.2 Case 2 Solution:

Similar to Case 1, the water temperature is raised by 20º C, but for this case the water temperature is initially at 20º C above the outside air temperature. As a result, for this case, $u^*(0,t^*)$ will range from 0.5 to 1, which will represent the same difference as 0 to 1 in the previous case. as a reult, $F'(\tau)$ is reduced by (1/2). If the losses, due to heat conduction through the wall were more significant, then $F'(\tau)$ would be further reduced, but as stated in section 4.1.3, the difference in time to heat within the full range of hot tub operation between no heat loss and maximum heat loss is less than 2%. As a result, the same heating time (9350 seconds) is used for this case. The temperature profile for times from 0.2 to 1.6 in steps of 0.2 is shown in Figure 6, below.

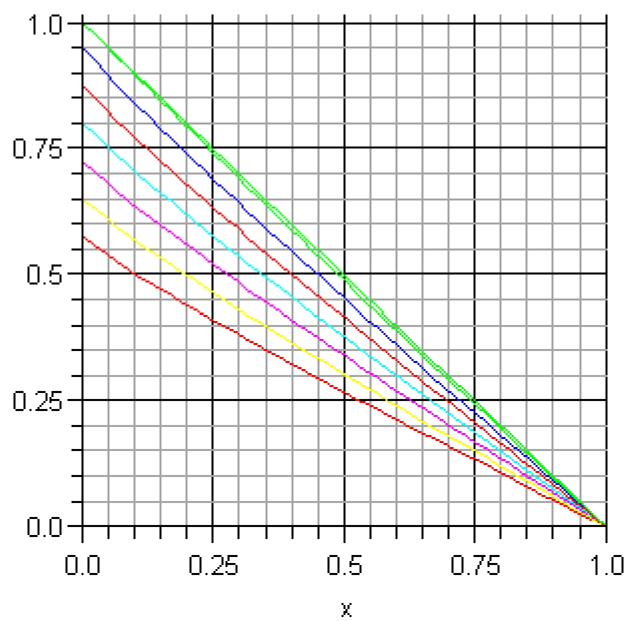


Figure 6: 2-D Plot of Analytical Solution for Case 2

Figure 6 shows that initial time results for case 2 are more linear than those achieved for case 1. Case 2 has because the initial contribution of the increase in temperature is small compared to the initial temperature for this case. A similar trend is expected for the case of cooling the hot tub due to the low level of heat loss. Looking further at the linearity of the temperature distribution for this case, u*(0.5, t*)/u*(0,t*) v. t* is shown in Figure 7 below:

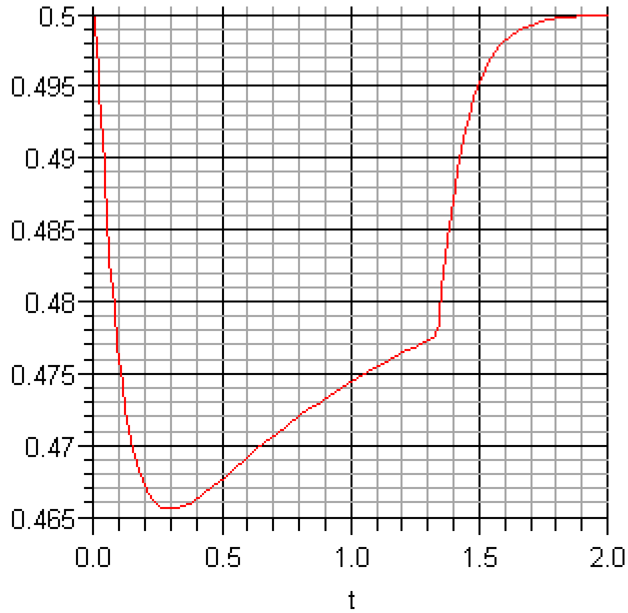


Figure 7: u*(0.5,t*)/u*(0,t*) v. t* for Analytical Solution of Case 2

Figure 7 shows that the ratio never drops below 0.45, for this case, while it was initially near 0 for case 1.

4.3 Temperature Distribution in Hot Tub Wall When Increasing Temperature – Numerical

4.3.1 Increasing Temperature Numerical Theory:

The one dimensional transient heat transfer through the hot tub walls is again considered with the heater on for the same conditions considered in section 4.2. The temperature distribution through the wall will be calculated using the explicit formulation of the finite difference method for the heat equation. Assuming a mesh of nodes across the x* and t* dimensions, with the letter i indicating the x nodal index and j indicating the t nodal index, the finite difference approximation of the heat equation is:

$$\frac{w_{i,j+1} - w_{i,j}}{\Delta t^*} = \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{\Delta x^{*2}} \tag{43}$$

The expression for explicit solution of interior nodes is:

$$w_{i,j+1} = (1 - \frac{2 \cdot \Delta t^*}{\Delta x^{*2}}) \cdot w_{i,j} + \frac{\Delta t^*}{\Delta x^{*2}} \cdot \left(w_{i+1,j} + w_{i-1,j}\right) \quad (44)$$

The initial condition and the condition for all t* at x* = 1 are set as constants at two of the boundaries. The final boundary condition is set as a linear function of time based on the small difference between the upper and lower bound solutions of time to heat. The final consideration for the explicit finite difference scheme is that the Courant-Friedrichs-Lewy (CFL) restriction (equation (45)), must be satisfied to obtain stable results.

$$\Delta t^* \le \frac{\Delta x^{*2}}{2} \quad (45)$$

4.3.2 Numerical Theory with boundary conditions based on Energy Balance

The boundary condition at the inner wall surface for all cases so far has been approximated as a linear function of time to allow for numerical solution in section 4.2 with Duhamel's method, and comparison to that answer in this section. Following numerical solution based on the linear boundary condition, the solution will be reevaluated numerically, using the energy balance equation as a boundary condition. For the water in the hot tub, the energy balance is:

$$\dot{Q}_{gen} = \rho_{water} \cdot V \cdot c_{p,water} \cdot \frac{\partial u}{\partial t} + k_{wall} \cdot A \cdot \left.\frac{\partial u}{\partial x}\right|_{x=0} \quad (46)$$

The dimensionless version is:

$$\frac{L^2 \cdot \dot{Q}_{gen}}{\rho_{water} \cdot V \cdot c_{p,water} \cdot \alpha_{wall}} = \frac{\partial u^*}{\partial t^*} + \frac{A \cdot L \cdot \rho_{wall} \cdot c_{p,wall}}{\rho_{water} \cdot V \cdot c_{p,water}} \cdot \left.\frac{\partial u^*}{\partial x^*}\right|_{x=0} \quad (47)$$

Converting the derivatives to the finite difference approximations, and substituting C1 and C2 for the constants, for the nodes along x*=0, the temperature for each node is calculated as:

$$w_{i,j+1} = C1 - C2 \cdot (w_{i,j} - w_{i+1,j}) + w_{i,j} \quad (48)$$

Where:

$$C1 = \frac{L^2 \cdot \dot{Q} \cdot \Delta t^*}{\rho_{water} \cdot V \cdot c_{p,water} \cdot \alpha_{wall}} \quad (49)$$

and

$$C2 = \frac{A \cdot L \cdot \rho_{wall} \cdot c_{p,wall} \cdot \Delta t^*}{\rho_{water} \cdot V \cdot c_{p,water} \cdot \alpha_{wall} \cdot \Delta x^*} \quad (50)$$

The results for this method will be compared to the result obtained with the linear boundary condition for validation.

4.3.3 Increasing Temperature Numerical Solution – Linear Boundary Condition:

The initial mesh was arbitrarily picked to have 51 nodes across the thickness of the wall, so Δx^* = 0.02. The maximum allowable time step for this case is 0.0002. The model was set up using Microsoft Excel computer software. A representative snapshot of the coding is shown below.

	Normalized explicit numerical method (heating)		
	dF/dt =	0.7523	
	k =	0.0002	
	h =	0.02	
t = \ x =	0	=B9+$C$5	=C9+$C$5
0	0	0	0
=A10+$C$4	=A11*$C$3	=(1-2*$C$4/($C$5^2))*C10+($C$4/($C$5^2))*(B10+D10)	=(1-2*$C$4/($C$5^2))*D10
=A11+$C$4	=A12*$C$3	=(1-2*$C$4/($C$5^2))*C11+($C$4/($C$5^2))*(B11+D11)	=(1-2*$C$4/($C$5^2))*D11
=A12+$C$4	=A13*$C$3	=(1-2*$C$4/($C$5^2))*C12+($C$4/($C$5^2))*(B12+D12)	=(1-2*$C$4/($C$5^2))*D12
=A13+$C$4	=A14*$C$3	=(1-2*$C$4/($C$5^2))*C13+($C$4/($C$5^2))*(B13+D13)	=(1-2*$C$4/($C$5^2))*D13
=A14+$C$4	=A15*$C$3	=(1-2*$C$4/($C$5^2))*C14+($C$4/($C$5^2))*(B14+D14)	=(1-2*$C$4/($C$5^2))*D14
=A15+$C$4	=A16*$C$3	=(1-2*$C$4/($C$5^2))*C15+($C$4/($C$5^2))*(B15+D15)	=(1-2*$C$4/($C$5^2))*D15
=A16+$C$4	=A17*$C$3	=(1-2*$C$4/($C$5^2))*C16+($C$4/($C$5^2))*(B16+D16)	=(1-2*$C$4/($C$5^2))*D16

Figure 8: Sample of Excel Finite Difference Coding.

The model was run from 0 to t* = 1.6. The temperature distribution across the wall is shown for values of t* from 0.2 to 1.6 in steps of 0.2.

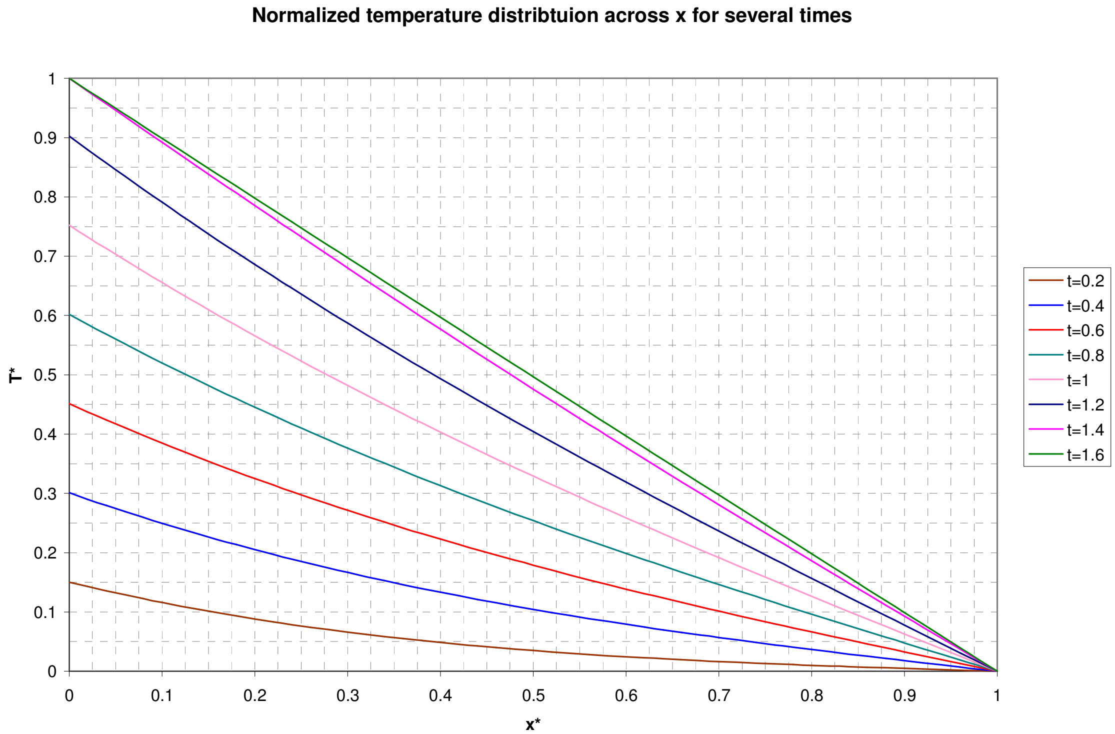


Figure 9: 2-D Plot of Numerical Solution for Case 1

As shown above, the temperature distribution starts out looking somewhat parabolic, and then as time goes on it approaches a linear temperature distribution. Figure 9 shows that the linear temperature distribution is not obtained for this case until after the steady state dimensionless temperature is reached. This will be further investigated by looking at the value of u* at x* = 0.5 and dividing it by u* at x* = 0 for all time, as shown in Figure 10. Since u* of x* = 1 is fixed at 0, the value of u*(0.5,t*) should approach 0.5 of u*(0,t*) as the temperature distribution becomes linear.

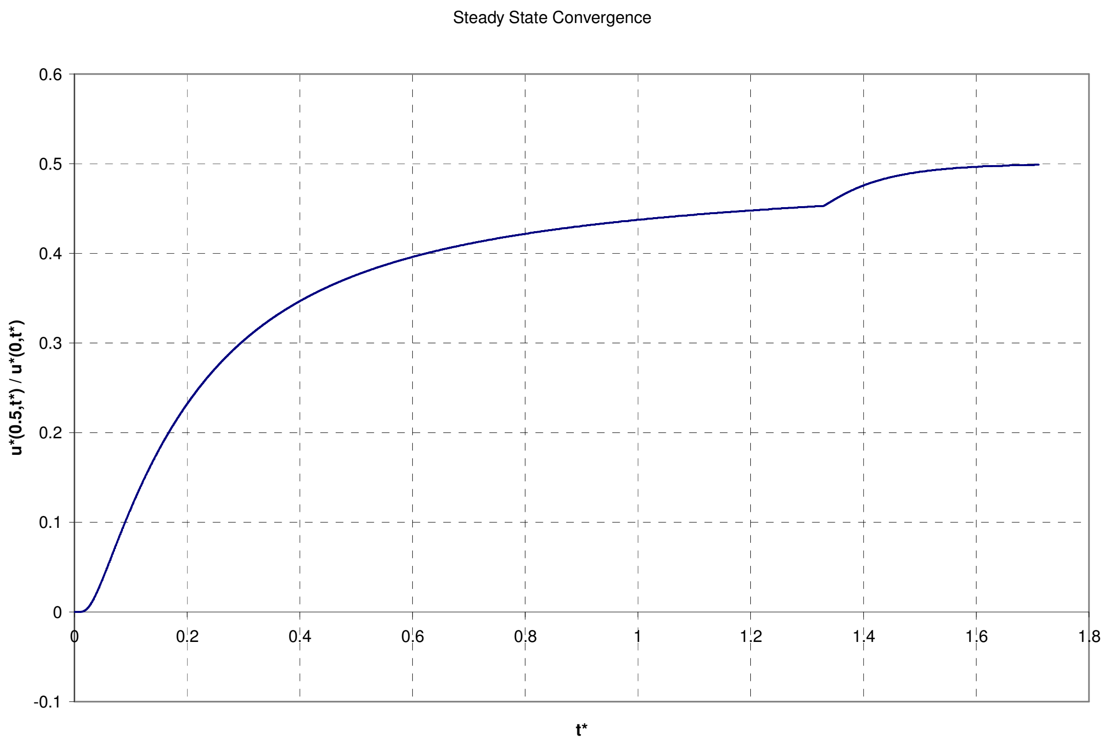


Figure 10: u*(0.5,t*)/u*(0,t*) v. t for Numerical Solution of Case 1

This parameter is used to examine the mesh dependance of the solution. Note that u*(0,t*) is just the plot of a line entered into excel, so if u*(0.5,t*)/u*(0,t*) is similar for cases, so too is u*(0.5,t*). The solution was performed with three difference amounts of nodes across the wall. The time step of 0.0002 was kept because it is the largest time step that satisifies the CFL restriction for $\Delta x^* = 0.02$.

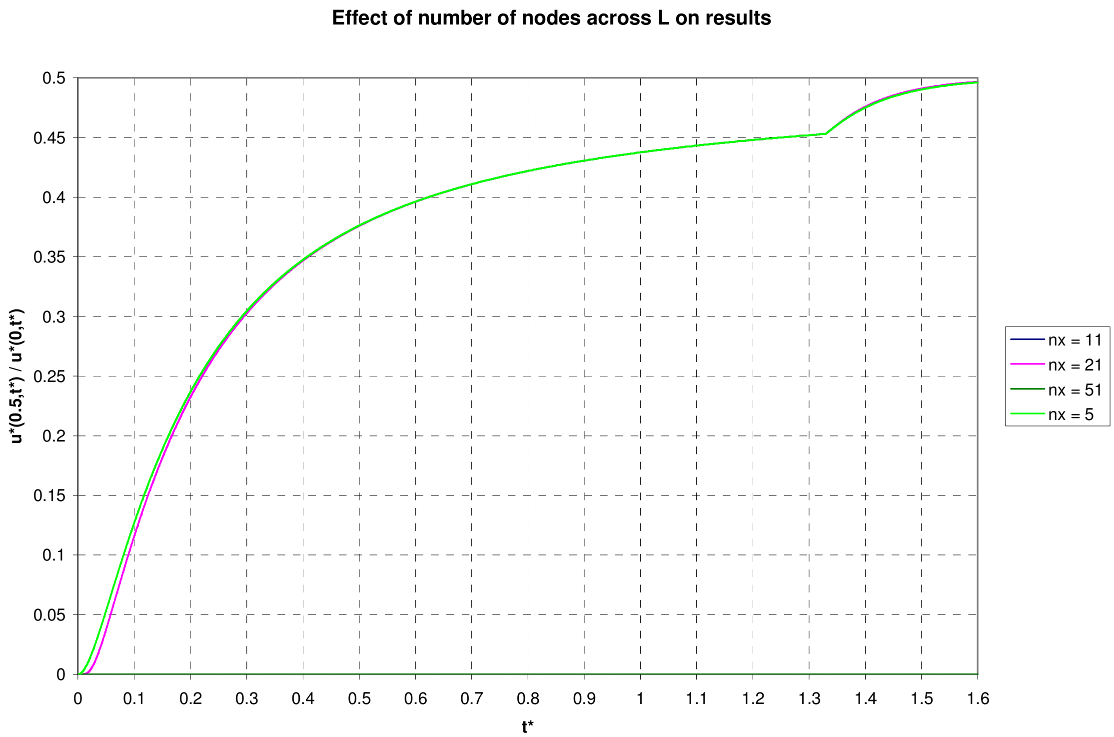


Figure 11: Effect of Number of x Nodes on Numerical Result of Case 1

For the most part, the lines lie right on top of each other, with the exception that the temperature rises more quickly for the solution with fewer nodes, but the solutions converge.

Next, the effect of the time step is investigated for nx = 11 (results appear equivalent to nx = 51, and larger time steps can be investigated). Figure 12 shows the results for three time steps:

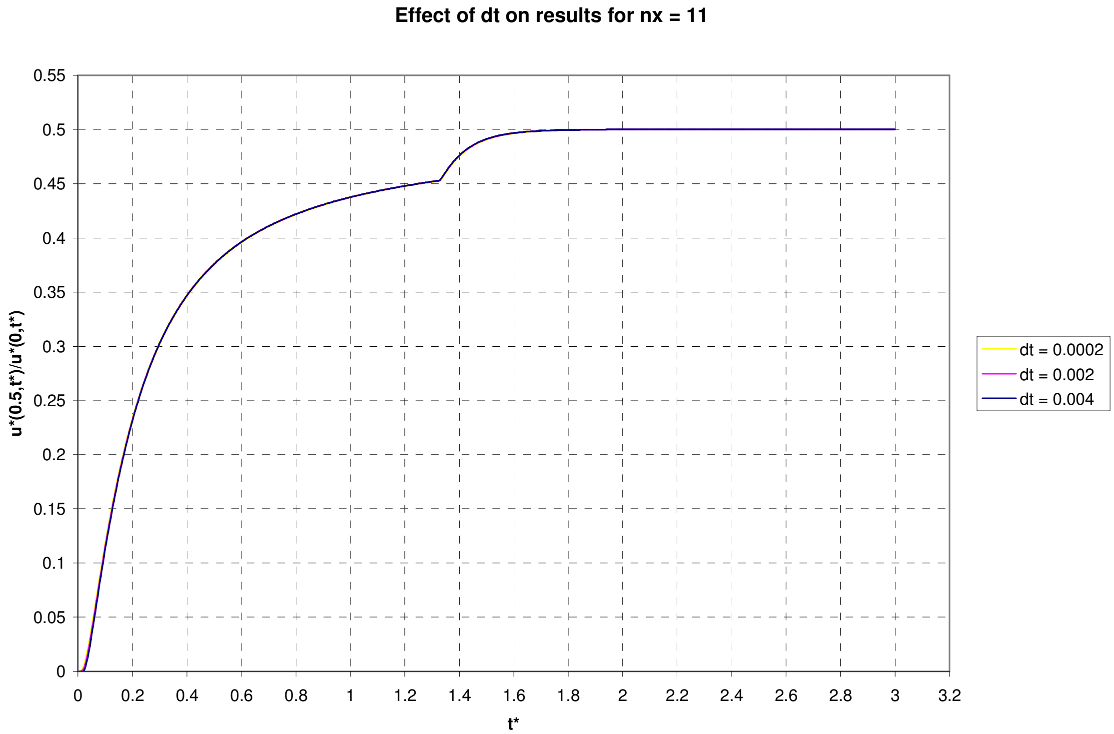


Figure 12: Effect of time step on Numerical Results of Case 1

Again the results appear to be very independent of the time step chosen. It should be noted however that a time step greater than 0.005 leads to an unstable result. The plot for a time step of 0.006 is shown below.

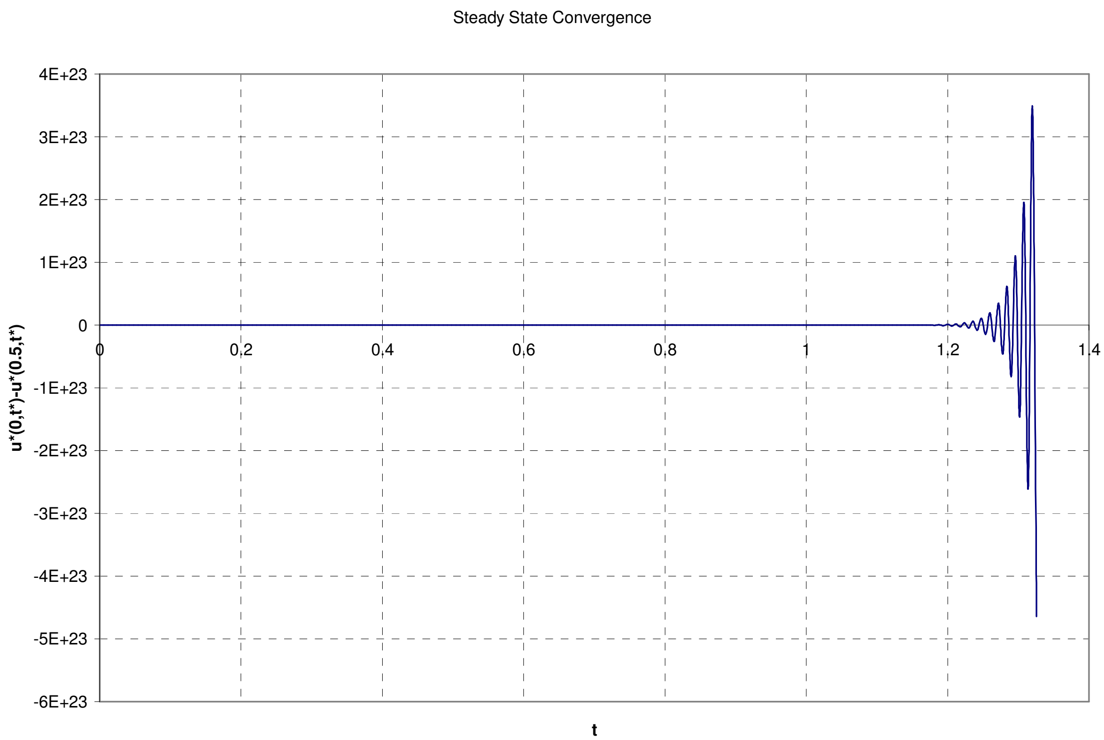


Figure 13: Result for Time Step That Does Not Satisfy CFL Restriction

The lack of change with different time steps and element meshes suggests that the results obtained are a good approximation. Further results will be calculated using the 11 node mesh across the wall thickness.

4.3.4 Increasing Temperature Numerical Solution – Energy Balance Boundary Condition:

The solution was obtained for the energy balance boundary condition at x* = 0 for nx = 11 and a time step size of 0.004, which as discussed above satisfies the CFL restriction. The dimensionless temperature distribution is shown in the Figure 14:

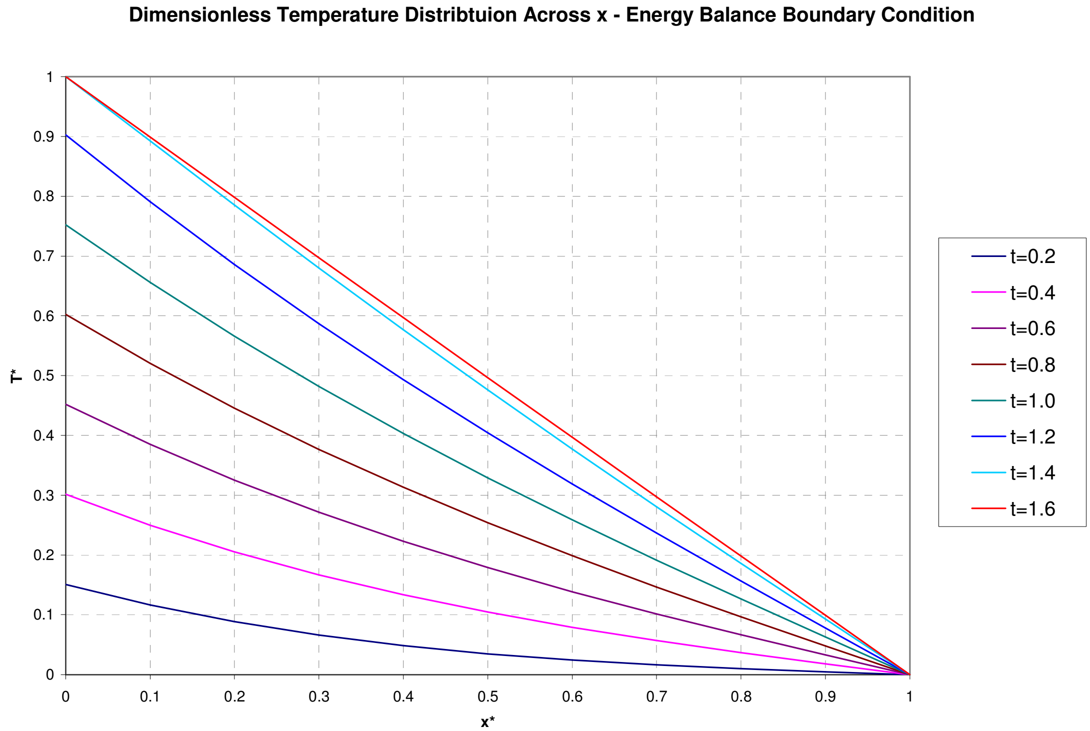


Figure 14: Results for Energy Based Boudary Condition – Case 1

As shown in the Figure 14, the results are very similar to those obtained with the linear boundary condition. The time (t*) to reach a dimensionless temperature of 1 is 1.332 (just picking the node at which 1 was exceeded and not interpolating back) vs 1.332 for the linear boundary condition. This corresponds to an actual heating time of 9368 seconds, with each step being equal to 28 seconds. So the total heating time is between 9340 seconds and 9368 seconds vs lower and upper bound heating times of 9323 and 9398 seconds respectively. The numerical results fall within the bounds and are therefore considered a good result (and I got my signs right).

4.3.5 Numerical Solution for Initial Temperature Difference with Steady State Heat Flux – Energy Balance Method:

The numerical solution to case 2 from the analytical section was obtained again using the energy balance boundary condition. A plot of the results is shown below:

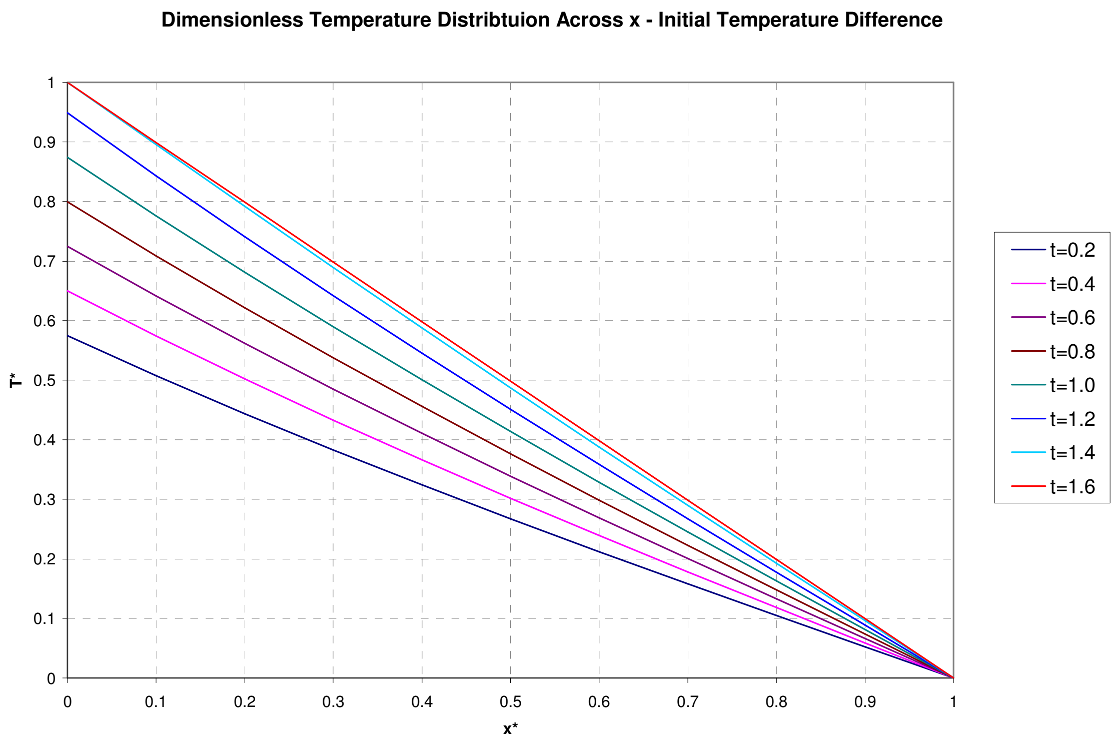


Figure 15: 2-D Plot of Numerical Results for Case 2.

As shown above, the results look similar to those obtained in the analytical case. The time to heat the water for this case was between 1.336 and 1.340, corresponding to actual times of 9396 and 9425. The lower and upper bounds for this case are 9323 and 9431 seconds respectively. Again, the solution falls within the expected bounds. The Linearity of the temperature profile is plotted against t* below:

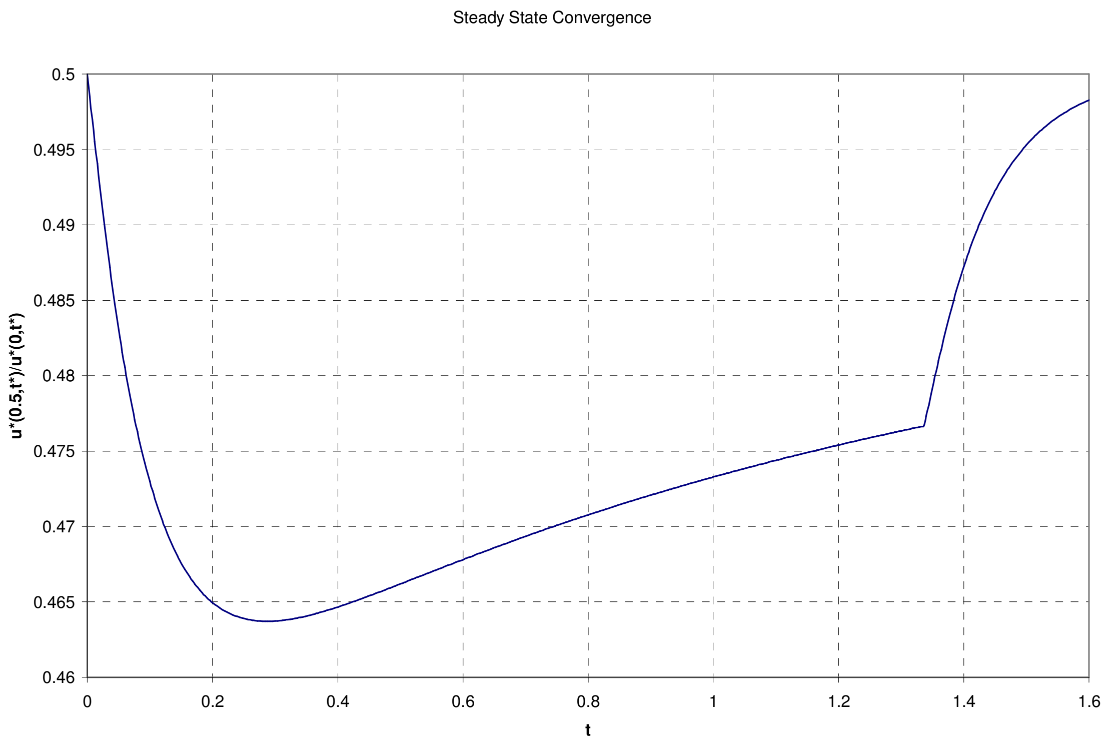


Figure 16: u*(0.5,t*)/u*(0,t*) v. t* for Numerical Solution of Case 2

As shown above, the temperature distribution across the wall remains nearly linear, as the ratio of u* at x = 0.5 vs 0 never falls below 0.46, and 0.5 indicates a linear temperature profile.

4.3.6 Heating the Hot Tub Conclusions:

The time to heat the hot tub by 20ºC with an initial temperature between 0 and 20ºC above outside air temperature has been bounded between 9323 and 9431 based on the cases of no heat loss and maximum heat loss.

The temperature distribution through the hot tub wall as the hot tub heats up has been calculated both analytically and numerically using an assumed linear temperature profile vs time at x = 0. The results of both methods are in agreement, and show that the temperature distribution through the wall as the inner surface heats up is not linear. There is a time lag between the temperature increase at the inner surface and the effect of that increase showing up through the wall. As the temperature increase becomes small relative to the magnitude of the wall temperature, the temperature distribution begins to resemble a steady state temperature distribution. This can be seen in the case of the tub being heated from an initial steady state temperature above the outside air. The results for that case show that the temperature distribution as the tub is heated to a new elevated temperature is nearly linear.

5. Cooling the hot tub:

The general assumption for this case is that the hot tub is heated such that the water is at a particular temperature and the temperature distribution through the walls is at steady state. The heater is turned off at time t = 0. The temperature of the water and through the wall is considered as a function of time.

5.1 Assumption that u*(x) across the wall remains linear for all t.

5.1.1 Theory – Cooling with continuously linear temperature gradient across wall

As a first simple case, the temperature distribution is assumed to remain linear as the temperature changes. The heat flux is assumed to be constant through the wall thickness, so only the difference between the inner and outer wall surfaces has to be considered. The outside wall surface temperature is fixed, while the inside wall surface temperature varies with the water temperature, so the equation of interest becomes:

$$\frac{k \cdot A \cdot (T - T_o)}{L} = \rho \cdot V \cdot c \cdot \frac{dT}{dt} \tag{51}$$

Which solves as:

$$T - T_o = (T_i - T_o)\exp\left(\frac{-k \cdot A}{L \cdot \rho \cdot V \cdot c} \cdot t\right) \tag{52}$$

Converting to the dimensionless parameters used to calculate heating times, this becomes:

$$u^* = \exp\left(\frac{-k \cdot A \cdot L}{\rho \cdot V \cdot c \cdot \alpha} \cdot t^*\right) \tag{53}$$

5.1.2 Results:

A plot of u* vs t* is shown below:

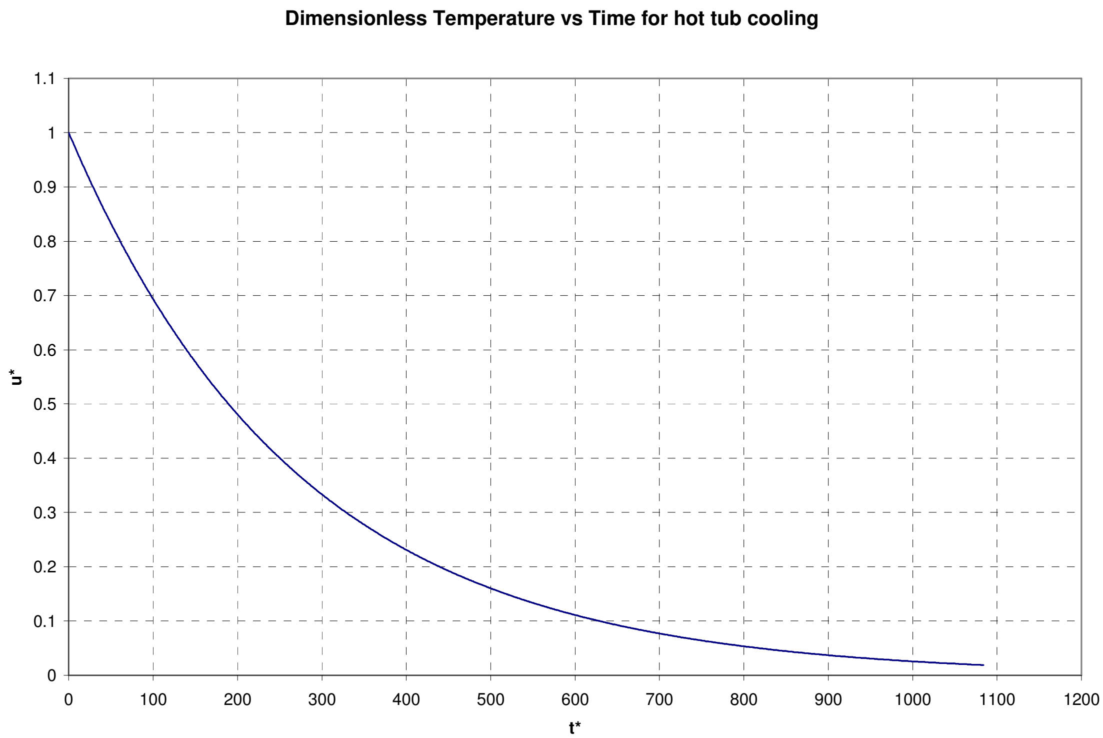


Figure 17: Cooling - u* v. t* for continuously linear temperature gradient across wall

As shown above, the time for the temperature to drop to 0.1 of its original value is 628, and time to drop to half of the original value is 189.3. These results are in dimensionless time, and the actual time associated with these values are 51.1 and 15.5 days

5.2 Numerical Solution of Wall Temperature Profile and time to cool:

5.2.1 Analytical Approach:

The analytical method based on Duhamel's method cannot be used in this case, because the boundary condition at x = 0 is non-linear and depends on both time and the temperature at x just greater than 0. As a result, a numerical result will be attempted.

5.2.2 Explicit Numerical Method:

In order to satisfy the CFL condition for 11 nodes across x, the maximum allowable time step is 0.005. The time to cool to 50% of the original value is 190. As a result, the minimum number of time steps required to get to 50% of the original temperature is 38,000. To get to 600 requires 120,000 steps, which is outside the range of Excel, so the result will obtained to verify the time at which the water temperature drops below 50% of the original value. A plot of the results obtained is shown below:

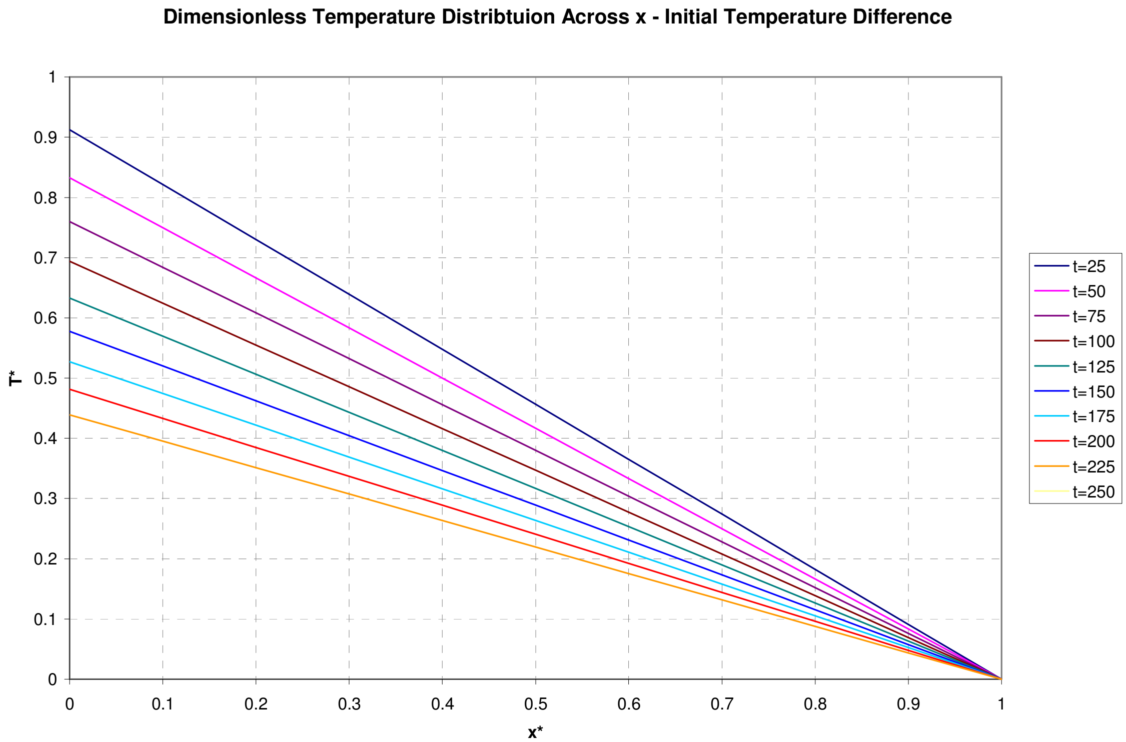


Figure 18: Results for Explicit Numerical Solution – Wall Temperature, Cooling

The above results show that the temperature profile across the wall remains very linear as the temperature of the water decreases. The time to reach $u^* = 0.5$ based on the numerical result is 189.51.

5.3 Conclusions – Cooling Hot Tub:

The numerical results calculated in section 5.2 support the assumption that the temperature gradient is constant or nearly so, as the water cools, as a result the method presented in section 5.1 provides accurate results. The time to cool the hot tub to half of its initial initial temperature relative to the outside air temperature is 15.5 days.

6. Discussion and Conclusions

This paper has calculated the maximum steady state temperature, heat fluxes at typical operating temperatures, time required to heat the hot tub and time for the hot tub temperature to drop from an initial temperature. The results were obtained based on a lumped capacitance model of the water, starting with the simplifying assumptions of no heat transfer, maximum heat transfer or a linear temperature profile through the walls of the hot tub. Analytical and Numerical analyses were used to obtain more refined results.

As the results show, due to the high steady state temperature of the hot tub relative to the typical operating range, all of the simple formulas intended to bound the problem provided results that were close to those for the more complex analyses. All results were checked against one another and were found to be in relative agreement.

The maximum steady state temperature that can be achieved based on the properties assumed and disregarding temperature effects (including melting) is 4120º C above the outside air temperature, as the tub is very well insulated. Heat flux out of the hot tub at 60º C (108 ºF) differential temperature, which is a conservative (high) estimate on the maximum differential temperature that would be maintained is 192 W. Even with a heater output of 13,200 W, it takes about 2.6 hours to raise the temperature of the water by 20ºC. Results also showed that the time to reduce the water temperature by 50% of some starting value above the outside temperature (calculated based on 40º C initial differential temperature) is about 15.5 days. The bottom line is that with the huge thermal capacitance that is a hot tub, nothing happens fast.

7. References

[1] Incropera F. P., and D. P. Dewitt: Fundamentals of Heat and Mass Transfer, 4th Ed., John Wiley and Sons, New York, 1996.

[2] Myers, G. E.: Analytical Methods in Conduction Heat Transfer. McGraw-Hill Book Company, New York, 1971

[3] Ozisik, M. N.: Heat Conduction. John Wiley and Sons, New York, 1980.

[4] Course Outline Notes http://www.rh.edu/~ernesto/S2006/AEM2/ Chapters 7 & 8

Appendix A: Properties of Hot Tub

Hot Tub Dimensions:

Outside Length = Outside Width = 1.5 m

Outside Height = 0.8 m

Wall Thickness = Cover Thickness = 0.05 m

Volume of Water = 1.47 m^3

Surface Area of Tub interior = 6.16 m^2

Properties of Water:

Density = 1000 kg/m^3

Specific Heat = 4186 J/kg*K

Properties of Wall / Cover:

Material assumed to be Urethane, two part mixture; rigid foam (properties from Ref [1]).

Density = 70 kg/m^3

Thermal Conductivity = 0.026 W/m*K

Specific Heat = 1045 J/kg*K

Thermal Diffusivity = 3.5543×10^{-7} m^2/s