Numerical simulation of the large elastic–plastic deformation behavior of hydrostatic stress-sensitive solids

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Received in final revised form 4 August 1999

Abstract

The present paper deals with the numerical simulation of the large elastic–plastic deformation and localization behavior of metals which are plastically dilatant and sensitive to hydrostatic stresses. The model is based on a generalized macroscopic theory taking into account macroscopic as well as microscopic experimental data obtained from tests with iron based metals. It shows that hydrostatic components may have a significant effect on the onset of localization and the associated deformation modes, and that they generally lead to a notable decrease in ductility. The continuum formulation relies on the mixed-variant metric transformation tensor which leads to the definition of an appropriate logarithmic strain measure. Its rate is additively decomposed into elastic and plastic as well as isochoric and volumetric strain rate tensors. Particular attention is focused on the formulation of a generalized $I_1 - J_2$ yield criterion to describe the effect of the hydrostatic stress on the plastic flow properties in metals. In contrast to classical theories of metal plasticity, the evolution of the plastic part of the strain rate tensor is determined by a non-associated flow rule based on a plastic potential function which is expressed in terms of stress invariants and kinematic parameters. Numerical analyses of the elastic–plastic deformation and localization behavior of hydrostatic stress-sensitive metals will demonstrate the influence of the constitutive description on critical strains as well as on localization behavior. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Elastic–plastic metals; Hydrostatic stress dependence; Plastic volume expansion; Logarithmic strains; Localization prediction; Finite element analyses


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PII: S0749-6419(99)00042-X
1. Introduction

An important problem in computational plasticity is the modelling and prediction of the macroscopic finite elastic–plastic deformation and localization behavior of metals subjected to complex loading conditions. As discussed, for example, by Hill (1950), and Khan and Huang (1995) macroscopically based continuum approaches usually start from experimental observations and elementary mechanical assumptions concerning the constitutive behavior. The overall macroscopic response of the solid is then obtained from suitable phenomenological models. Most numerical simulations in metal plasticity are then based on isochoric theories using von Mises’ $J_2$ yield condition associated with plastically non-dilatant normality flow rule.

Generally, experimental results obtained from simple uniaxial tension and compression tests tend to confirm the von Mises yield condition, but experimental studies on the effect of superimposed hydrostatic pressure on the deformation behavior of iron based materials (Spitzig et al., 1975, 1976) as well as of aluminum (Spitzig and Richmond, 1984) have shown that the flow stress depends approximately linearly on hydrostatic pressure. In particular, the yield and flow stresses increase with hydrostatic pressure, and it is generally recognized that the flow stresses of some metals are larger in uniaxial compression than in uniaxial tension. This phenomenon — known as the strength-differential (SD)-effect — has been studied extensively in recent years (Casey and Sullivan, 1985), and it is now well established that the magnitude of the SD-effect increases with the amount of carbon in the solution of the ferrite matrix of high-strength steels, and that it decreases with increasing tempering temperature. For example, Spitzig et al. (1975, 1976) examined the effect of the hydrostatic pressure on the yield and flow stresses in uniaxial tension and compression, and they showed experimentally that tests performed under high hydrostatic pressure lead to rising stress–strain curves.

The extensive tests of Spitzig et al. (1975, 1976) on the behavior of high-strength metals undergoing uniaxial tension and compression, on the other hand, indicate that plastic deformation may also be accompanied by a permanent volume expansion and that the latter is largely insensitive to the sign of the mean stress. The classical theory of plastic flow of metals assumes that any pressure dependence of yielding is associated with irreversible dilatation through the normality flow rule. But their experiments, which were conducted over a sufficiently large pressure range, cast some doubt on the general validity of the normality rule. In particular, they show that the permanent volume expansion is similar in tension and compression and, furthermore, is observed to be an approximately linear function of strain as long as small plastic strains occur. The measured values of the plastic volume increase in high-strength steels are at least an order of magnitude less than the numerically predicted values obtained from continuum plasticity normality concept. In aluminum, on the other hand, the plastic dilatancy measured in tension and compression tests may be seen to be negligible. Thus, when predictions were based on the normality rule in combination with an experimentally determined hydrostatic stress-dependent macroscopic yield condition, noticeably too large values of the plastic volume change were obtained than in the experiments. This suggests that the
normality concept may not always be appropriate to describe the large deformation behavior of metals.

The experimental observations discussed above lead to the assumption that the increase in flow stress with hydrostatic pressure is a result of the effect of pressure on the basic flow events, such as dislocation motion with respect to the microstructure of metal crystals whereas the small plastic volume changes are consistent with expected changes in dislocation density. This suggests that metals are, to some extent, frictional materials like granular solids but to a much smaller degree. Thus, the appropriate continuum plasticity model may deviate from the normality flow rule. As a consequence the classical plasticity theory for metals undergoing large elastic–plastic deformations under complex loading conditions has to be modified to be able to take into account the physical effects discussed above, see e.g. Casey and Sullivan (1985).

These effects are generally ignored in computational metal plasticity — possibly due to the difficulties involved in obtaining the necessary material properties as well as to the added complexities in the mechanical model and its numerical implementation. The availability of powerful computers and efficient numerical methods make it possible, however, to carry out extensive numerical simulations and thus render the second objection less and less critical. They also open up many new opportunities for the development of physically sound plasticity models for metals (Aifantis, 1987; Sellers and Douglas, 1990) and are of great value in gaining a deeper understanding of the finite elastic–plastic deformation as well as the localization and failure behavior of solids. However, previous finite element analyses on the effect of superimposed hydrostatic pressure on the finite elastic-plastic deformation behavior of crystalline solids have been presented by Brünig (1997, 1998) and Brünig and Obrecht (1998). Their formulations take into account deviations from the classical Schmid-rule of the critical resolved shear stress in the microscale as well as microscopic plastic volume changes including a macroscopic non-associated flow rule. Preliminary numerical simulations demonstrate the influence of additional constitutive and kinematic parameters on plastic yielding and permanent volume expansion in tension and compression tests. Furthermore, Li and Richmond (1997) stated that in the hardening regime the non-normality approach provides a strong destabilizing effect in metal polycrystals, whereas Lee and Ghosh (1996) discussed the influence of the third deviatoric stress invariant on the prediction of shear bands.

Thus, the present paper deals with the numerical simulation of the large elastic–plastic deformation and localization behavior of metals which are plastically dilatant and sensitive to hydrostatic stresses. The model is based on a generalized macroscopic theory taking into account macroscopic as well as microscopic experimental data obtained from tests with iron based metals. It shows that hydrostatic components may have a very pronounced effect on the onset of localization and the associated deformation modes, and that they generally lead to a notable decrease in ductility.

The continuum formulation relies on a multiplicative decomposition of the mixed-variant metric transformation tensor into elastic and plastic parts. This leads to the definition of an appropriate logarithmic strain measure whose rate is shown to be additively decomposed into elastic and plastic parts. The mixed-variant logarithmic
elastic strain tensor is then employed to define a local isotropic hyperelastic constitutive law. This leads to a generalized Hooke’s law for large strain analyses which has been shown to be in good agreement with experimental results. Particular attention is focused on the formulation of a generalized $I_1 - J_2$ yield criterion to describe the effect of hydrostatic stress on the plastic flow properties of metals. In contrast to classical theories of metal plasticity the evolution of the plastic part of the logarithmic strain rate tensor is determined by a non-associated flow rule based on a plastic potential function which is expressed in terms of stress invariants and kinematic parameters. The use of logarithmic strain tensors leads to geometrically exact additive decompositions into decoupled volumetric (spheric) and isochoric (deviatoric) plastic strain rates which is an essential aspect in the formulation of irreversible volume expanding phenomena.

The computation of the components of the logarithmic strain tensors is performed in a very general and efficient manner using first and higher order Padé approximations rather than more familiar transformations to principal strain space. Furthermore, estimates of the local stress and strain histories are obtained via a highly stable and very accurate explicit scalar integration procedure which was developed from Nemat-Nasser’s (1991) plastic predictor method. The algorithm for the evolution of corresponding tensorial quantities — especially the current elastic and plastic metric tensors — employs an integration scheme with an exponential shift. The finite element procedure is based on the principle of virtual work and associated linearized variational equations are obtained using a consistent linearization algorithm. The appearance of macroscopic shear bands is obtained from the solution of a local bifurcation problem and continuing localized deformations as well as post-failure predictions are computed via a global post-bifurcation analysis.

Numerical simulations of the elastic–plastic deformation behavior of plastically dilatant and hydrostatic stress-sensitive metals will show the physical effects described above and also the efficiency of the formulation. Their results are in excellent agreement with experimental data. Furthermore, a variety of large strain elastic–plastic problems involving severe localization phenomena will be presented and the influence of various model parameters on the deformation and localization behavior of hydrostatic stress-sensitive metals will be discussed.

2. Fundamental governing equations

Most mechanical elastic–plastic formulations presented so far rest on the fundamental ideas of Lee (1969) who used a multiplicative decomposition of the material deformation gradient into an elastic and a plastic part. Based on the advantages of this product decomposition Simo (1988) established a framework for finite strain elastic-plastic constitutive equations using a hyperelastic material model formulated in terms of Cauchy–Green tensors. On the other hand, Weber and Anand (1990), Eterovic and Bathe (1990), Perić et al. (1992), and Sansour and Kollmann (1997) developed material models for finite deformation theories using the logarithmic Hencky strain tensor.
Moreover, an alternative description of large strain elastic–plastic theory is based on the mixed-variant metric transformation tensor introduced by Lehmann (1982a,b) which will be multiplicatively decomposed into a plastic and an elastic part. This framework avoids a priori the introduction of plastic rotations which are controversially discussed in plasticity literature. Thus, it stands in contrast to the mechanical formulations based on the multiplicative decomposition of the deformation gradient discussed above. Based on the advantages of Lehmann’s idea Brüning (1999) used the metric transformation tensors to formulate mixed-variant logarithmic strains. This approach will be employed in this paper and extended systematically for a consistent definition of hyperelastic stress–strain relations and the representation of non-associated mixed-variant flow rules.

In particular, the kinematic theory for the mechanics of large deformations of solids is based on the metric transformation tensor

\[ \mathbf{Q} = Q^j_i \otimes \mathbf{g}^j = \hat{G}^{ik} G_{kl} \mathbf{g}_i \otimes \mathbf{g}^j = \hat{\mathbf{G}}^{-1} \mathbf{C}. \]  \hspace{1cm} (1)

In Eq. (1) \( \mathbf{g}_i \) and \( \mathbf{g}^j \) denote the covariant and contravariant base vectors of the initial configuration, respectively, \( G_{ij} \) and \( \hat{G}_{ij} \) represent the metric coefficients of the base vectors of the current and initial configurations, and \( \mathbf{C} \) and \( \mathbf{G} \) are the associated tensors with respect to the undeformed initial configuration. This leads to the definition of arbitrary strain tensors of the Seth family, and as a particular case the mixed variant logarithmic Hencky strain tensor

\[ \mathbf{H} = \frac{1}{2} \ln \mathbf{Q} = \frac{1}{2} (\ln Q)^j_i \mathbf{g}_i \otimes \mathbf{g}^j = H^j_i \mathbf{g}_i \otimes \mathbf{g}^j \]  \hspace{1cm} (2)

will be formulated as an isotropic tensor function of the metric transformation tensor (1). The spherical part of the logarithmic strain tensor

\[ \text{tr} \mathbf{H} = \mathbf{H} \cdot \mathbf{1} = \ln \frac{dv}{dv_0} = \ln J \]  \hspace{1cm} (3)

exactly describes the volume change of the solid and, thus, corresponds to the logarithm of the Jacobian \( J = \sqrt{\det \mathbf{C}} \). Then, the deviatoric part of the Hencky strain tensor

\[ \text{dev} \mathbf{H} = \mathbf{H} - \frac{1}{3} \text{tr} \mathbf{H} \mathbf{1} \]  \hspace{1cm} (4)

exactly represents the isochoric deformation.

Following Brüning (1999) the macroscopic formulation of the large deformation behavior of elastic–plastic materials is based on the multiplicative decomposition of the metric transformation tensor \( \mathbf{Q} \) into its respective plastic and elastic parts

\[ \mathbf{Q} = \mathbf{Q}^{pl} \mathbf{Q}^{el} \]  \hspace{1cm} (5)
with
\[ Q_{pl} = G^{ik} G_{kj} \hat{g}_i \otimes \hat{g}_j \]  
and
\[ Q_{el} = G^{ik} G_{kj} \hat{g}_i \otimes \hat{g}_j. \]

In Eqs. (6) and (7) $G_{ij}$ denotes the metric tensor of the base vectors of an intermediate configuration which represents a fictitious unstressed state at fixed values of the internal variables. To be able to describe reversible deformations the elastic strain tensor
\[ H_{el} = \frac{1}{2} \ln Q_{el} \]  
is introduced. These strains are assumed to be kinematically independent from accompanying inelastic deformations.

Furthermore, the rates of the total and elastic strain tensors (2) and (8)
\[ \dot{H} = \frac{1}{2} (\ln Q) = \frac{1}{2} Q^{-1} \dot{Q} \]  
and
\[ \dot{H}_{el} = \frac{1}{2} (\ln Q_{el}) = \frac{1}{2} Q^{-1}_{el} \dot{Q}^e_{el} \]  
will be considered. Taking into account the multiplicative decomposition (5) Eq. (9) leads to the additive decomposition of the total Hencky strain rate
\[ \dot{H} = \frac{1}{2} Q^{el-1} Q^{pl-1} \dot{Q}^{el} Q_{el} + \frac{1}{2} Q^{el-1} \dot{Q}^{el} = \dot{H}^{pl} + \dot{H}^{el} \]  
into its elastic (10) and plastic parts using the definition
\[ \dot{H}^{pl} = \frac{1}{2} Q^{-1} Q^{pl} Q_{el}, \]

which will be employed to formulate the inelastic constitutive behavior.

Using, now, the logarithmic Hencky strain tensor $H$ and its work-conjugate counterpart, the mixed-variant stress tensor
\[ T = T^{ij}_{\beta} \hat{g}_i \otimes \hat{g}_j, \]
the rate of specific mechanical work \( \dot{\omega} \) is defined by

\[
\rho_o \dot{\omega} = \mathbf{T} \cdot \dot{\mathbf{H}}
\]  

(14)

where \( \rho_o \) denotes the initial mass density (see e.g. Brünig, 1999). The rate of mechanical work (14) can be additively decomposed according to

\[
\rho_o \dot{\omega} = \rho_o \dot{\omega}^{\text{el}} + \rho_o \dot{\omega}^{\text{pl}} = \mathbf{T} \cdot \dot{\mathbf{H}}^{\text{el}} + \mathbf{T} \cdot \dot{\mathbf{H}}^{\text{pl}}
\]  

(15)

into an elastic part \( \dot{\omega}^{\text{el}} \) governed by thermodynamic state equations as well as into a plastic part \( \dot{\omega}^{\text{pl}} \). The formulation of inelastic constitutive equations is based on the introduction of phenomenological variables. These scalar and tensor-valued variables may be seen as a basis to carry forward informations from the microscale (e.g. crystal lattice) to the phenomenological macroscale. Thus, the internal variables determine the hardening state of the material. It should be noted that interrelations between the mixed-variant stress tensor \( \mathbf{T} \) [Eq. (13)] as well as other well-known stress tensors often used in continuum mechanics are given by Lehmann (1982b) and \( \mathbf{T} \) is physically equivalent to the well-known Kirchhoff stress tensor in the sense that their invariants coincide.

Moreover, the specific free energy function \( \phi \) is introduced and is assumed to be additively decomposed into an elastic and a plastic part

\[
\phi = \phi^{\text{el}} + \phi^{\text{pl}}
\]  

(16)

which will be expressible as functions of thermodynamic state variables. Thus, the elastic part \( \phi^{\text{el}}(\dot{\mathbf{H}}^{\text{el}}) \) may be formulated in terms of the elastic strain tensor whereas the plastic part \( \phi^{\text{pl}} \) takes into account scalar-valued as well as tensor-valued internal state variables.

In this context, the second law of thermodynamics (Clausius–Duhem inequality) for isothermal processes can be written in the form

\[
\dot{\omega} - \dot{\phi} \geq 0.
\]  

(17)

Making use of Eqs. (15) and (16) one arrives at

\[
\mathbf{T} \cdot \dot{\mathbf{H}}^{\text{el}} + \mathbf{T} \cdot \dot{\mathbf{H}}^{\text{pl}} - \rho_o \frac{\partial \phi^{\text{el}}}{\partial \mathbf{H}^{\text{el}}} \cdot \dot{\mathbf{H}}^{\text{el}} - \rho_o \dot{\phi}^{\text{pl}} \geq 0
\]  

(18)

or

\[
\left( \mathbf{T} - \rho_o \frac{\partial \phi^{\text{el}}}{\partial \mathbf{H}^{\text{el}}} \right) \cdot \dot{\mathbf{H}}^{\text{el}} + \mathbf{T} \cdot \dot{\mathbf{H}}^{\text{pl}} - \rho_o \dot{\phi}^{\text{pl}} \geq 0.
\]  

(19)

Now, standard arguments concerning non-dissipative processes in the elastic range lead to the potential relation for the mixed-variant stress tensor (thermic state equation)
\[ T = \rho_0 \frac{\partial \phi^{cl}}{\partial \mathbf{H}^{el}} \]  

and, assuming Eq. (20) to hold in the plastic range as well, Eq. (19) reduces to the so-called intrinsic dissipation inequality

\[ D = T \cdot \dot{\mathbf{H}}^{pl} - \rho_0 \dot{\phi}^{pl} \geq 0, \]  

where \( D \) represents the dissipation function. Thus the evolution equation for the plastic part of the deformation will be formulated in terms of \( \dot{\mathbf{H}}^{pl} \) defined by Eq. (12).

3. Constitutive equations

The finite elastic part of the isotropic material behavior is assumed to be governed by the Helmholtz free energy function

\[ \rho_0 \phi^{cl} = \mu \text{dev}\mathbf{H}^{el} \cdot \text{dev}\mathbf{H}^{el} + \frac{1}{2} K (\text{tr}\mathbf{H}^{el})^2 \]  

where \( \mu \) and \( K \) represent the shear and bulk modulus of the material, respectively. It admits an additive decomposition of the elastic Hencky strain tensor

\[ \mathbf{H}^{el} = \text{dev}\mathbf{H}^{el} + \frac{1}{3} \text{tr}\mathbf{H}^{el} \mathbf{1} \]  

into the elastic bulk strain measure \( \text{tr}\mathbf{H}^{el} \) computing the elastic dilatation as well as the deviator \( \text{dev}\mathbf{H}^{el} \) representing isochoric elastic deformations. Taking into account the hyperelastic constitutive relationship (20) the mixed-variant stress tensor may be expressed in the form

\[ \mathbf{T} = 2\mu \text{dev}\mathbf{H}^{el} + K \text{tr}\mathbf{H}^{el} \mathbf{1} \]  

If any effect of plastic straining on the finite elastic material behavior is neglected, the associated tensor of elastic moduli may be determined from

\[ \mathbf{C} = \rho_0 \frac{\partial^2 \phi^{cl}}{\partial \mathbf{H}^{el} \otimes \partial \mathbf{H}^{el}} = 2\mu \mathbf{I}^D + K \mathbf{1} \otimes \mathbf{1} \]  

with

\[ \mathbf{I}^D = \left( \delta^j_k \delta^l_j - \frac{1}{3} \delta^l_k \right) \mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}_l \otimes \mathbf{g}^k \]  

and the second order identity tensor \( \mathbf{1} \).

In order to characterize the hydrostatic stress-dependent plastic material a phenomenological \( I_1 - J_2 \) flow theory including isotropic nonlinear work-hardening is
taken into account. Experimental results on aluminum and high-strength steels (Spitzig et al., 1975, 1976; Spitzig and Richmond, 1984) showed that the yield and flow stresses are adequately described by the relation

\[ aI_1 + \sqrt{J_2} = c \]  

(27)

where \( I_1 = \text{tr}T \) and \( J_2 = \frac{1}{2} \text{dev}T \cdot \text{dev}T \) are invariants of the stress tensor (13), \( a \) represents the hydrostatic stress coefficient, and \( c \) means the strength coefficient. These coefficients, \( a \) and \( c \), are strain-dependent, whereas experimental data have shown that the ratio \( \alpha = a/c \) is constant and, therefore, like the elastic constants, a property of the bulk iron lattice. Therefore, the \( I_1 - J_2 \) yield condition (27) can be written in the form

\[ f^\text{pl}(I_1, J_2) = \sqrt{J_2} - c(1 - \alpha I_1) = 0. \]  

(28)

In addition, the consistency condition

\[ \dot{f}^\text{pl} = \left( \frac{1}{2 \sqrt{J_2}} \text{dev}T + a \mathbf{1} \right) \cdot \dot{T} - \dot{c}(1 - \alpha I_1) = 0 \]  

(29)

ensures that the subsequent yield condition remains satisfied during any incremental deformation.

It should be noted that Eq. (28) represents an extended version of the well-known Drucker–Prager (1952) yield condition which is often used to describe inelastic material behavior of soils but in their case both \( a \) and \( c \) were chosen to be constant and no work-hardening effects are taken into account.

To be able to compute the inelastic part of the strain tensor the plastic potential function has to be formulated:

\[ g^\text{pl} = bI_1 + \sqrt{J_2} \]  

(30)

where \( b \) represents the plastic dilatation coefficient. This leads to the flow law

\[ \dot{\mathbf{H}}^\text{pl} = \dot{\lambda} \frac{\partial g^\text{pl}}{\partial \mathbf{T}} = \dot{\lambda} \left( \frac{1}{2 \sqrt{J_2}} \text{dev}T + b \mathbf{1} \right) \]  

(31)

with the non-negative scalar-valued factor \( \dot{\lambda} \). Since the kinematic parameter \( b \) and the constitutive parameter \( a \) appearing in Eq. (27) do not coincide the flow law (31) is not associated to the yield condition (27) and, hence, does not fulfill the normality rule generally assumed in metal plasticity.

Furthermore, the deviatoric mixed-variant orientation tensor

\[ \mathbf{N} = \frac{1}{\sqrt{2J_2}} \text{dev}T \]  

(32)
is introduced which satisfies the conditions \( \mathbf{N} \cdot \mathbf{N} = 1 \) and \( \mathbf{N} \cdot \dot{\mathbf{N}} = 0 \). This leads to the definition of the isochoric equivalent plastic strain rate

\[
\dot{\gamma}_{\text{iso}} = \mathbf{N} \cdot \dot{\mathbf{H}}^{\text{pl}} = \frac{1}{\sqrt{2}} \dot{\lambda}
\]  

(33)

which will be used to express the plastic strain rate tensor (31) in the form

\[
\dot{\mathbf{H}}^{\text{pl}} = \dot{\gamma}_{\text{iso}} \mathbf{N} + \dot{\gamma}_{\text{iso}} \sqrt{2} b \mathbf{I} = \text{dev} \dot{\mathbf{H}}^{\text{pl}} + \frac{1}{3} \text{tr} \dot{\mathbf{H}}^{\text{pl}} \mathbf{I}.
\]  

(34)

Thus, it is possible to identify isochoric plastic as well as dilatant plastic strain rates

\[
\text{dev} \dot{\mathbf{H}}^{\text{pl}} = \dot{\gamma}_{\text{iso}} \mathbf{N}
\]

(35)

and

\[
\text{tr} \dot{\mathbf{H}}^{\text{pl}} = 3\sqrt{2} b \dot{\gamma}_{\text{iso}},
\]

(36)

respectively. Using, now, Eqs. (29), (32), (24), and (11) one arrives at the following scalar-valued rate constitutive equation

\[
(1 - \alpha I_1) \dot{c} = \sqrt{2} \mu \dot{\varepsilon}_{\text{iso}} + 3a K \dot{\varepsilon}_{\text{vol}} - \left( \sqrt{2} \mu + 9 \sqrt{2} ab K \right) \dot{\gamma}_{\text{iso}}
\]

(37)

with the equivalent total isochoric and volumetric strain rates

\[
\dot{\varepsilon}_{\text{iso}} = \mathbf{N} \cdot \text{dev} \dot{\mathbf{H}}
\]

(38)

and

\[
\dot{\varepsilon}_{\text{vol}} = \text{tr} \dot{\mathbf{H}}
\]

(39)

4. Numerical integration of the constitutive equations

In general, efficient and stable iterative techniques are employed to solve the discretized equilibrium equations for each time step in displacement-based finite element procedures for nonlinear problems. The results of each iteration may be seen as estimates of the incremental displacements which are used to compute the current stress state and other field variables in the integration points of the finite elements. Accordingly, these variables are known at time \( t_n = t \) as well as the estimate of the total metric transformation tensor \( \mathbf{Q} \) at time \( t_{n+1} = t + \Delta t \). Then, the problem is to integrate the evolution equations for \( \mathbf{Q}^{\text{pl}} \) and \( c \) across the time increment \( \Delta t \) in order
to calculate the current values of the stress and strain tensors. This is performed via a stable, accurate and efficient integration algorithm which is well suited for finite element analyses involving large elastic-plastic deformations.

In computational plasticity approaches, many research groups employ the radial return technique (elastic predictor-plastic corrector method) to integrate the constitutive rate equations. This procedure works well for smooth yield conditions and associated flow rules as long as time increments remain reasonably small. Since the trade-off between accuracy and computational efficiency is an issue of current interest, Nemat-Nasser (1991) presented an alternative algorithm. His plastic predictor–elastic corrector technique yields nearly the exact solution even in large time steps for elastic-plastic materials. In addition, Brünig et al. (1995) and Brünig (1999) presented finite element analyses for finite elastic–plastic strain problems and their finite element calculations permit remarkably large load increments with almost no loss in accuracy and requires only very few iterations to achieve converged equilibrium solutions at the global level.

In particular, since Eq. (37) describes the entire rate constitutive behavior of the material for arbitrary loading paths, it is possible to derive the complete stress and deformation history from a straightforward numerical integration of (37). This integration is performed in two stages using first a plastic predictor step which is followed by an elastic corrector together with a standard midpoint rule. Thus, when \( \hat{t} \) is taken as a general evolution parameter and the scalar plastic strain rate \( \dot{\gamma} \) is assumed to be constant during the interval \( t \leq \hat{t} \leq t + \Delta t \) Eq. (37) leads to

\[
\left[ 1 - \alpha f(\hat{t}) \right] [c(t + \Delta t) - c(t)] = \sqrt{2} \mu (\Delta \varepsilon - \Delta \gamma)
\]  

(40)

where the increment of the equivalent strain is given by

\[
\Delta \varepsilon = \Delta \varepsilon_{\text{iso}} + \frac{3aK}{\sqrt{2} \mu} \Delta \varepsilon_{\text{vol}}
\]

(41)

with its respective isochoric

\[
\Delta \varepsilon_{\text{iso}} = \int_{t}^{t+\Delta t} \mathbf{N} \cdot \dot{\mathbf{H}} d\Theta
\]

(42)

and volumetric parts

\[
\Delta \varepsilon_{\text{vol}} = \int_{t}^{t+\Delta t} \mathbf{1} \cdot \dot{\mathbf{H}} d\Theta,
\]

(43)

whereas the increment of the equivalent plastic strain can be expressed in the form

\[
\Delta \gamma = \left( 1 + \frac{9abK}{\mu} \right) \Delta \gamma_{\text{iso}}.
\]

(44)
In the plastic predictor step the entire deformation increment is taken to be plastic
\[ \Delta \gamma_p = \Delta \epsilon \] (45)
and, as a result, the associated predictor strength coefficient at the end of the interval
\[ c_p(t + \Delta t) = c(\gamma(t) + \Delta \epsilon) \] (46)
is assumed to be determined from the equivalent stress–equivalent plastic strain relationship of the material. Clearly, this assumption leads to an overestimation of both the equivalent plastic strain increment \( \Delta \gamma \) and the current equivalent stress measure \( c(t + \delta t) \). The respective errors are then given by
\[ \Delta_{cr} c = c_p(t + \Delta t) - c(t + \Delta t) \] (47)
\[ \dot{\gamma}_{cr} (\dot{\gamma}) = \dot{\epsilon} (\dot{\gamma}) - \dot{\gamma} (\dot{\gamma}) = \frac{1 - \alpha I_1 (\dot{\gamma})}{\sqrt{2\mu}} \dot{\gamma}, \] (48)
and
\[ \Delta_{cr} \gamma = \int_{t}^{t+\Delta t} \dot{\gamma}_{cr} d\Theta = \dot{\gamma}^{*}_{cr} \Delta t. \] (49)

Furthermore, it is assumed that the errors in strength coefficient and equivalent plastic strain are related by the plastic constitutive relationship
\[ \Delta_{cr} c \cong H \Delta_{cr} \gamma \] (50)
where the hardening modulus \( H = \partial c / \partial \gamma \) is computed at the end of the plastic predictor increment. Using, now, Eqs. (40), (47), (48), and (50) one obtains the following estimate of the error in the equivalent plastic strain
\[ \Delta_{cr} \gamma = \left[ \frac{\sqrt{2\mu}}{1 - \alpha I_1} + H \right]^{-1} \left( c_p(t + \Delta t) - c(t) \right), \] (51)
and then the respective total plastic strain and strength coefficient at the end of the time increment are given by
\[ \gamma (t + \Delta t) = \gamma (t) + \Delta \gamma \] (52)
with
\[ \Delta \gamma = \Delta \gamma_p - \Delta_{cr} \gamma \] (53)
and
\[
c(t + \Delta t) = c(t) + \frac{\sqrt{2} \mu}{1 - \alpha I_1} \Delta e_{\gamma}.
\]

(54)

Taking into account Eq. (44) the isochoric part of the equivalent plastic strain increment is given by
\[
\Delta \gamma_{\text{iso}} = \left(1 + \frac{9abK}{\mu} \right)^{-1} \Delta \gamma.
\]

(55)

Corresponding estimates of the respective tensorial quantities may now be derived using the above fundamental relationships. In particular, the incremental plastic strain tensor is given by
\[
\Delta H^{\text{pl}} = \text{dev} \Delta H^{\text{pl}} + \frac{1}{3} \text{tr} \Delta H^{\text{pl}} \mathbf{I}
\]

(56)

where
\[
\text{dev} \Delta H^{\text{pl}} = \Delta \gamma_{\text{iso}} \hat{N}
\]

(57)

represents the deviator of the incremental plastic strain tensor and the incremental plastic dilatation may be computed from
\[
\text{tr} \Delta H^{\text{pl}} = 3\sqrt{2} b \Delta \gamma_{\text{iso}}.
\]

(58)

In Eq. (57) \( \hat{N} \) denotes the mean deviatoric orientation tensor (normalized deviatoric stress tensor) over the time increment \( t \leq \tilde{t} \leq t + \Delta t \) given by
\[
\hat{N} = \frac{1}{2} (N(t + \Delta t) + N(t)).
\]

(59)

Taking into account the incremental elastic law
\[
\text{dev} \Delta T = \text{dev} T(t + \Delta t) - \text{dev} T(t) = 2\mu \text{dev} \left( \Delta H^{\text{el}} \right)
\]

\[
= 2\mu \left( \text{dev} \Delta H - \text{dev} \Delta H^{\text{pl}} \right)
\]

(60)

and making use of Eqs. (46), (47) and (29) the current unit normal
\[
N(t + \Delta t) = \frac{\left( \sqrt{2J_2(t) - \mu \Delta \gamma_{\text{iso}}} \right) N(t) + 2\mu \text{dev}(\Delta H)}{\sqrt{2J_2(t + \Delta t) + \mu \Delta \gamma_{\text{iso}}}}
\]

(61)
is also known. Therefore, all tensorial quantities can now be computed without any iteration. In particular, the elastic strain increment

$$\Delta H^e = \Delta H - \Delta H^p = \frac{1}{2} \int_t^{t+\Delta t} (\ln Q^e) \, d\Theta$$

leads to the logarithm of the current elastic metric transformation tensor

$$\ln Q^e(t + \Delta t) = \ln Q^e(t) + 2 \Delta H^e.$$

Using Eq. (63) the current elastic logarithmic strain tensor (8) is then calculated by

$$H^e(t + \Delta t) = \frac{1}{2} \ln Q^e(t + \Delta t) = H^e(t) + \Delta H^e.$$

Finally, the respective current stress tensor $T(t + \Delta t)$ follows from the hyperelastic constitutive relationship (24), while Eq. (25) yields the associated tensor of current elastic moduli $C(t + \Delta t)$.

The current plastic metric transformation tensor may then be computed from

$$Q^p(t + \Delta t) = Q(t + \Delta t) Q^{e^{-1}}(t + \Delta t),$$

where the elastic metric transformation is given by

$$Q^e(t + \Delta t) = \exp(2H^e).$$

5. Analysis of plastic flow localization

At a certain stage in the loading process, the initially smooth displacement distribution may develop into a pattern which involves highly localized deformations. This may lead to the formation of cracks and to rapid fracture at overall strains which are only slightly larger than those which correspond to the onset of localization. In general, the values of these critical strains are quite sensitive to the constitutive description of the material, and it is also well known that classical elastoplastic theories based on smooth yield surfaces and the normality flow rule may not be able to predict localization at all. Material models which differ from these classical ones, on the other hand, such as those which include dilatational plastic effects as well as deviations from the normality rule, may be better suited to predict localization at realistic levels of strain (Rudnicki and Rice, 1975; Lee and Ghosh, 1996). In addition, bifurcation analyses give a good indication of the ability of various constitutive descriptions to realistically predict the elastic-plastic deformation and failure behavior of materials and structures.
Fundamentals of the analysis of localization in elastic-plastic solids have been developed by Hill (1962) and Rice (1977), and its main purpose is to determine the conditions under which bifurcation into a localized mode can occur. Briefly, the conditions of compatibility inside and outside the localized band, rate equilibrium across the band interfaces as well as associated constitutive relations lead to the fundamental equation governing the localization process, see Brüning (1996) for details. Its nontrivial solution determines the onset of localization whereas the initial band orientations are computed using the eigenvalues of the corresponding acoustic tensor. This local bifurcation analysis is performed at all integration points of the finite elements, whereas the subsequent development of the localized deformation pattern is again obtained from a global numerical analysis.

6. Computational aspects and finite element implementation

To be able to compute the components of the mixed-variant logarithmic strain tensor $H = \frac{1}{2} \ln Q$ in Eq. (2), Brüning (1999) proposed the use of first and higher order Padé approximations. In particular, taking into account the abbreviation

$$A = (Q - 1)(Q + 1)^{-1}$$

(67)

it is possible to express the first Padé approximation of $H$ by

$$H_{(1)} = A$$

(68)

the 2nd Padé approximation by

$$H_{(2)} = A \left(1 - \frac{1}{3} A^2\right)^{-1}$$

(69)

the 3rd Padé approximation by

$$H_{(3)} = \left(A - \frac{4}{15} A^3\right) \left(1 - \frac{3}{5} A^2\right)^{-1}$$

(70)

and the 4th Padé approximation by

$$H_{(4)} = \left(A - \frac{11}{21} A^3\right) \left(1 - \frac{6}{7} A^2 + \frac{3}{35} A^4\right)^{-1}$$

(71)

respectively. In a large number of numerical simulations the first Padé approximation (68) is an excellent approximation of the true value of the logarithm as long as the principal metric tensor components remain moderate. But for larger principal stretches the use of higher order Padé approximations is necessary to be able to
compute of the components of the Hencky strain tensor in an accurate manner. Limits for the application of the respective Padé approximations (68)–(71) have been examined in some detail by Brüning (1999). The materials considered in this paper are characterized by small elastic strains compared to the total strains. Therefore, the elastic strain tensor is accurately computed by the first Padé approximation whereas the total strain components may require higher order Padé approximations. Similarly, the tensor exponent $\mathbf{Q}^{el}(t + \Delta t) = \exp(2\mathbf{H}^{el})$ in Eq. (66) will be determined by the first Padé approximation

$$
\mathbf{Q}^{el}_{(1)} = (\mathbf{I} - \mathbf{H}^{el})^{-1}(\mathbf{I} + \mathbf{H}^{el})
$$

which leads to a sufficiently accurate value of the elastic exponential function.

Moreover, in order to preserve the quadratic rate of asymptotic convergence in the performance of Newton’s iterative solution algorithm the above constitutive technique is employed in conjunction with the consistent current tensor of elastic-plastic moduli

$$
\mathbf{C}^{ep} = \frac{d\mathbf{T}}{d\mathbf{H}},
$$

In particular, taking into account the current tensor of elastic moduli (25) the differential formulation of Eq. (24) is given by

$$
d\mathbf{T} = \mathbf{C} d\mathbf{H}^{el}
$$

where — using integration of Eq. (11) — the differential of the elastic strain tensor is obtained from

$$
d\mathbf{H}^{el} = d\Delta \mathbf{H}^{el} = d\Delta \mathbf{H} - d\Delta \mathbf{H}^{pl}
$$

and with (56)–(58) one gets

$$
d\Delta \mathbf{H}^{pl} = \left(\mathbf{\hat{N}} + \sqrt{2b}\mathbf{I}\right)d\mathbf{\Delta} \mathbf{\gamma}_{iso} + \frac{1}{2}d\mathbf{\Delta} \mathbf{\gamma}_{iso}d\mathbf{N},
$$

$$
d\Delta \mathbf{\gamma}_{iso} = \left(1 + \frac{9abk}{\mu}\right)^{-1}d\mathbf{\Delta} \mathbf{\gamma}.
$$

Furthermore, Eqs. (41), (44), (46), (51) and (52) lead to the differential of the scalar valued plastic strain increment

$$
d\mathbf{\Delta} \mathbf{\gamma} = \frac{\sqrt{2}\mu}{\sqrt{2\mu + (1 - aI_1)H}}(d\mathbf{N}:d\mathbf{\Delta} \mathbf{H} + \mathbf{N}d\mathbf{\Delta} \mathbf{H}) + \frac{3ak}{2\mu + (1 - aI_1)H}d\mathbf{\Delta} \mathbf{H}.
$$
In addition, Eq. (32) yields
\[ dN = \frac{-1}{\sqrt{2}J_2} \text{dev} T d\sqrt{J_2} + \frac{1}{\sqrt{2}J_2} d(\text{dev} T) \] (79)
with
\[ d\sqrt{J_2} = \frac{1}{\sqrt{2}} N \cdot dT. \] (80)

Substituting Eq. (80) into Eq. (79) then leads to
\[ dN = \frac{-1}{\sqrt{2}J_2} (N \otimes N) dT + \frac{1}{\sqrt{2}J_2} \Pi^p dT \] (81)

where \( \Pi^p \) is given by Eq. (26). Finally, Eqs. (74)–(81) furnish the following final representation of the tensor of actual elastic–plastic moduli
\[
C^{sp} = \left[ 1 + \frac{\mu \Delta \gamma}{\sqrt{2}J_2} (\Pi^p - N \otimes N) \right]^{-1} \left[ C - \left( 1 + \frac{9abK}{\mu} \right)^{-1} \frac{1}{\sqrt{2} \mu + (1 - \alpha I_1)H} \right. \\
\left. \left( 2 \sqrt{2} \mu^2 \hat{N} \otimes N + 6aK\mu \hat{N} \otimes I + 6bK\mu I \otimes N + 9 \sqrt{2}abK^2 I \otimes I \right) \right]. \] (82)

Note that these moduli depend on the constitutive equations as well as the time integration procedure used to obtain the current field quantities.

Following Brüning (1999) the finite element procedure is based on the principle of virtual work
\[ \delta \Pi(\delta u, u) = \int_{B_0} \delta H \cdot T dv_0 - \int_{\partial B_0} \delta u \cdot \hat{t}_0 \, da_0 = 0 \] (83)

where \( B_0 \) and \( \partial B_0 \) denote the volume and surface of the body in the initial configuration. The first integral in Eq. (83) represents the variation of the current stored energy density while the second accounts for the contribution of the prescribed surface tractions \( \hat{t}_0 \).

Using a consistent linearization procedure, choosing suitable shape functions and nodal degrees of freedom for the unknown displacements, carrying out the appropriate integrations and finally assembling the individual element stiffness matrices and load vectors, one arrives — as usual — at a set of linearized algebraic equations for the nodal displacement increments, which may be written in the familiar abbreviated form
\[ K_T \Delta V = R. \]  

(84)

In Eq. (84), \( K_T \) denotes the global tangent stiffness matrix (which depends explicitly on the current elastic-plastic moduli as well as the current state of stress and deformation), \( R \) corresponds to the residual unbalanced force vector, and \( \Delta V \) represents the vector of incremental displacements. Eq. (84) is solved recursively until a suitable norm of \( \Delta V \) falls below a sufficiently small limit, thus indicating that a converged equilibrium solution has been reached.

7. Numerical simulations

7.1. Uniaxial tension and compression tests

Various uniaxial tension and compression tests on iron-based materials and aluminum presented by Spitzig et al. (1975, 1976) and Spitzig and Richmond (1984) have shown that the flow stress depends non-linearly on the plastic strain. Therefore, the current strength coefficient \( c \) appearing in the generalized \( I_1 - J_2 \) yield condition (27) will be numerically simulated by the power law

\[ c = c_0 \left( \frac{H_0 \gamma}{nc_0} + 1 \right)^n. \]  

(85)

In Eq. (85) \( c_0 \) means the initial yield strength, \( H_0 \) represents the initial hardening parameter and \( n \) denotes the hardening exponent. Fig. 1, for example, shows the current yield strength-equivalent plastic strain curve for 4330 steel. As can be seen, the chosen material properties \( c_0 = 854 \) MPa, \( H_0 = 25000 \) MPa and \( n = 0.07 \) lead to good agreement with the experimental data presented by Spitzig et al. (1975). In addition, the power law (85) will be employed to adequately describe the current yield strength in aluminum tested by Spitzig and Richmond (1984). The respective material properties are then \( c_0 = 14.82 \) MPa, \( H_0 = 4000 \) MPa and \( n = 0.26 \).

Some iron-based materials, however, show strong saturation behavior which cannot be adequately described by the power law (85). For example, the current yield strength coefficient of aged maraging steel will be numerically simulated by the saturation law

\[ c = (c_S - c_0) \tanh \left( \frac{H_0 \gamma}{c_S - c_0} \right) + c_0, \]  

(86)

where \( c_S \) denotes the saturation flow strength. The comparison of experimental data presented by Spitzig et al. (1976) and the corresponding numerical calculation of the \( c - \gamma \) curve with \( c_0 = 870 \) MPa, \( H_0 = 96000 \) MPa and \( c_S = 1120 \) MPa is given in Fig. 2 which shows good agreement with test results.

In addition, the experiments have shown that the flow stress depends nearly linearly on the hydrostatic stress which will be described by the \( I_1 - J_2 \) yield criterion.
(28) including the hydrostatic stress coefficient $\alpha$. In particular, the effect of hydrostatic stress on the deformation characteristics of aluminum in uniaxial tension tests was to increase the flow stress. But there is no apparent difference between the absolute values of the tension and compression yield or flow stresses which indicates an approximately zero SD. The experimental data presented by Spitzig and Richmond (1984) can be numerically simulated by $\alpha = 56 \, \text{TPa}^{-1}$. Steels, on the other hand, are characterized by remarkable SD-effects and testing under superimposed
hydrostatic pressure raises the stress-strain curves. It has been shown that the hydrostatic stress-sensitive material behavior of steels and other iron-based materials can be numerically simulated by the hydrostatic stress coefficient $\alpha = 20 \text{TPa}^{-1}$ in an accurate manner.

Furthermore, tension and compression tests with aged maraging steel as well as 4310 and 4330 steels, which were conducted over a small strain range, have shown remarkable irreversible volume expansions which are approximately proportional to the plastic strain. The underlying physical process is assumed to be an increase in dislocation density as long as crystallographic deformations occur. Therefore, the increase in permanent volume expansion will be limited and the kinematic parameter $b$ appearing in the flow rule (31) is computed from the volumetric plastic strain $\gamma_{\text{vol}}$ which in the moderately large plastic strain range is assumed to be adequately described by the saturation function

$$\gamma_{\text{vol}} = \gamma_{\text{vol},S} \tan h(10\gamma).$$

(87)

The plastic volume increase during plastic deformation is shown in Fig. 3 and numerical results computed for moderate strains are compared to experimental data within the small strain range. In particular, the saturation values are assumed to be $\gamma_{\text{vol},S} = 4.8 \times 10^{-4}$ in aged maraging steel and $\gamma_{\text{vol},S} = 6.5 \times 10^{-4}$ in 4310 and 4330 steels. Analyses of the experimental data of aluminum as well as of unaged maraging and austenitic stainless steel — which are characterized by relatively small SD-effects — on the other hand, show that the volume expansions resulting from plastic deformations are very small and could actually be less than the test data because they represent the minimum value capable of being obtained from the density determi-
nation due to the accuracy of the scale readings. Therefore, the plastic volume expansions in these materials will be neglected in numerical simulations presented here. These data support the hypothesis that the SD-effect may be seen as a consequence of the irreversible volume expansion which occurs during plastic deformation.

7.2. Tension and compression tests under plane strain conditions

The numerical analyses discussed in this section deal with the finite deformation behavior and bifurcation predictions of uniaxially loaded geometrically perfect rectangular specimens with free ends. Numerical calculations take into account plane strain conditions and are carried out using crossed-triangle elements. The elastic material behavior of 4330 and aged maraging steel is based on Young’s modulus \( E = 200,000 \) MPa and Poisson’s ratio \( \nu = 0.3 \) whereas the rate-independent plastic material properties are described by the nonlinearly work-hardening behavior discussed above.

First, the elastic–plastic deformation and bifurcation behavior of 4330 steel will be studied in some detail. In particular, if the onset of plastic yielding as well as the subsequent plastic flow is assumed to be described by the \( J_2 \) yield condition [\( \alpha = 0 \) in Eq. (28)] no bifurcation is predicted numerically even in largely strained specimens as can be seen from Fig. 4. But with increasing hydrostatic stress parameter \( \alpha \) the largely strained specimens tend to localize. For example, if the \( I_1 - J_2 \) yield condition (28) with \( \alpha = 20 \) TPa\(^{-1} \) is taken into account, which is the material parameter for steels based on experimental observations and measurements discussed above, the acoustic tensor remains zero at the critical logarithmic longitudinal strain \( H_{11}^{\text{crit}} = 19\% \). In addition, corresponding load-deflection curves are shown in Fig. 5. The numerical calculation based on the \( J_2 \) yield condition (\( \alpha = 0 \)) leads to maximum in tensile load max\( P \) at the engineering strain \( u/l = 0.09 \) and with increasing deformation a small decrease in load and no bifurcation is observed. The numerical simulation taking into account the \( I_1 - J_2 \) yield condition with the hydrostatic stress parameter \( \alpha = 20 \) TPa\(^{-1} \), however, leads to nearly 5% smaller loads in the elastic-plastic range with load maximum at \( u/l = 0.08 \). Again, with increasing global deformation a small decrease in load occurs, but at \( u/l = 0.21 \) the bifurcation point is computed. Therefore, numerical simulations taking into account the experimentally observed effect of hydrostatic stress on the current flow properties of 4330 steel predict the bifurcation at moderately large logarithmic strain levels whereas the neglect of hydrostatic stress dependencies does not show any bifurcation point.

Moreover, the elastic–plastic deformation behavior and bifurcation predictions of aged maraging steel will be discussed. In general, the value of the critical strain is quite sensitive to the constitutive description of the material and it is often claimed that classical elastic–plastic theories based on \( J_2 \) yield surfaces and the normality flow rule may not be able to predict bifurcation. Surprisingly, the numerical simulation of the tension test with aged maraging steel based on the \( J_2 \) yield criterion and the associated flow rule predicts bifurcation at the longitudinal logarithmic strain \( H_{11}^{\text{crit}} = 3.4\% \). The zero determinant of the acoustic tensor at this remarkably small critical strain value may be seen as a consequence of the rapidly saturated hardening
behavior observed in the experiments and taken into account in the numerical calculations, see Fig. 2. In addition, Fig. 6 shows the decrease in critical logarithmic strain with increasing hydrostatic stress parameter $\alpha$ and, thus, the realistic material model taking into account the effect of hydrostatic stress on the current plastic flow properties leads to very early bifurcation in aged maraging steels. In particular, with $\alpha = 20 \text{ TPa}^{-1}$ which is based on the experiments discussed above bifurcation is predicted at the critical longitudinal strain $H_{11}^{\text{crit}} = 1.8\%$. Furthermore, corresponding load-deflection curves are given in Fig. 7. The numerical simulation based on the $J_2$ yield criterion ($\alpha = 0$) leads to the maximum load $\max P$ at the elongation $u/l = 0.019$.
and with further increasing deformation a decrease in load is observed. Bifurcation takes place at \( u/l = 0.038 \). The numerical simulation taking into account the hydrostatic stress effects, on the other hand, predicts nearly 16% smaller loads in the elastic-plastic range with load maximum at \( u/l = 0.015 \) and bifurcation at \( u/l = 0.018 \). Again, these numerical results based on the experimentally verified \( I_1 - J_2 \) yield condition seem to be more realistic than those based on hydrostatic stress-independent \( J_2 \) yield criteria.

In addition, the influence of the kinematic parameter \( b \) appearing in the plastic potential function \( (30) \) on the elastic-plastic deformation behavior and bifurcation prediction has been studied. Various numerical calculations have shown that this parameter \( b \) neither remarkably affects the load-deflection curves in tension and compression tests under plane strain conditions nor the prediction of the bifurcation. The independence of the numerical results on the kinematic parameter \( b \) may be seen as a consequence of the small plastic volume expansion as well as the assumption that they have been taken to be limited in the moderately large plastic strain range and, thus, have been described by the saturation function discussed above. The earlier bifurcation predicted in numerical simulations of uniaxially loaded tension and compression tests taking into account the generalized \( I_1 - J_2 \) yield condition \( (28) \) may be seen primarily as a result of deviations from the normality flow rule.

7.3. Rectangular specimens with fixed straight ends

The numerical analyses discussed above on the finite deformation behavior of geometrically perfect bars with free end boundary conditions have shown that the application of hydrostatic stress-sensitive material models in metal plasticity and non-associated flow rules remarkably influence the onset of localization (bifurcation points). Therefore, the numerical simulations presented in this section will analyse
the influence of material models on subsequent localization behavior. Rectangular bars with fixed straight ends are uniaxially loaded in tension tests taking into account plane strain conditions.

In particular, slender metal workpieces with thickness $h = 1$ and length $l = 4$. undergoing uniaxial tension tests will be analysed numerically. The corresponding finite element mesh consists of 1600 (20 × 80) crossed-triangle elements. For simplicity and due to the symmetry conditions involved only a quarter of the specimen has been treated. Numerical simulations of the elastic-plastic deformation behavior of bars formed with aged maraging steel based on the generalized hydrostatic stress-sensitive $I_1 - J_2$ yield criterion (28) predict the onset of localization at the global elongation $u/l = 0.016$ immediately after maximum load has been passed. Primary bifurcation into shear bands occurs in the center region of the bar and the corresponding inclination of the shear band angle is $43^\circ$ with respect to the loading direction. Fig. 8a clearly shows that with increasing elongation values small shear bands occur which may be seen as a precursor of ductile fracture in this region. Numerical calculations which neglect the effect of hydrostatic stress on the flow characteristics of metals, on the other hand, compute the onset of localization at $u/l = 0.020$. Again, primary bifurcation into shear bands occur in the center region of the specimen and the inclination of the shear band angle is $43^\circ$. Then, with increasing elongation shear bands spread over wider areas, and the global deformation behavior is slightly superposed by a diffuse neck, see Fig. 8b.
Fig. 8. Metal specimens: deformed configurations.

Superposed necking, however, becomes more and more dominant when highly ductile metals are considered. Numerical simulations of tension tests of 4330 steel bars based on the hydrostatic stress-sensitive material model discussed above show the onset of localization at \( u/l = 0.057 \) and the corresponding shear band inclination is \( 33^\circ \) with respect to the loading axis. Here, further spread of shear bands during continuing elongation is largely prevented by the development of a dominant neck, see Fig. 8c. Numerical calculations which deal with the deformation behavior of 4330 steel specimens based on hydrostatic stress-independent \( J_2 \) yield condition, on the other hand, lead to necking behavior shown in Fig. 8d. It shows no localized shear band development at all.
In summary, numerical simulations taking into account the experimentally observed effect of hydrostatic stress on the current flow properties of iron based materials predict that the onset of localization occurs at lower strain levels and lead to more distinct deformation modes. In addition, aluminum bars have been analysed numerically and in uniaxial tension tests they do not show localized or diffuse modes but only smooth deformation behavior. This may be seen as a consequence of the early plastic yielding with remarkable subsequent work-hardening characteristics.

8. Conclusions

A nonlinear finite element model for the macroscopic rate-independent analysis of hydrostatic stress-sensitive metals has been presented. The continuum formulation is based on the multiplicative decomposition of the mixed-variant metric transformation tensor. This leads to the additive split of the rate of appropriate logarithmic strain tensors into elastic and plastic as well as into isochoric and volumetric parts. Particular attention has been focused on the formulation of the generalized $I_1 - J_2$ yield condition to be able to simulate numerically the effect of hydrostatic stress on the plastic flow properties in metals. The emergence of plastic deformations is accompanied by inelastic volume expansion which has been seen as a consequence of increase in dislocation density. Therefore, it is numerically simulated by an adequate saturation function. Hence, in contrast to classical theories of metal plasticity the evolution of the inelastic logarithmic strains is determined by a non-associated flow rule.

In the numerical analysis higher order Padé approximations are used to compute the components of the logarithmic strain tensors. The integration of the scalar valued constitutive equation is based on the generalized plastic predictor algorithm. Corresponding current tensorial quantities are then obtained by an integration scheme with an exponential shift.

The numerical calculations dealing with the finite deformation behavior of steel specimens undergoing uniaxial tension experiments have shown that the application of hydrostatic stress-sensitive material models and non-associated flow rules remarkably influence the prediction of the onset of localization. The proposed model for the plastic volume increase, on the other hand, neither remarkably effects the load-deflection curves nor the prediction of bifurcation points. Furthermore, hydrostatic stress-dependent material models have a remarkable influence on the subsequent localization behavior. They lead to more distinct and smaller shear bands in tension tests and, generally also to a notable decrease in ductility. By contrast, numerical calculations based on hydrostatic stress-independent plasticity theories show the development of a dominant superposed neck. Since plastic dilatancy is limited in high-strength metals undergoing finite strains the earlier onset of localization as well as the subsequent localized deformation modes predicted in numerical simulations of uniaxially loaded tension tests may be primarily seen as the consequence of deviations from the normality flow rule.
References


