Lateral buckling of web-tapered I-beams: A new theory

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Abstract

This paper presents a new theory for the lateral buckling of web-tapered I-beams. Linear analysis is first conducted by taking account into the tapering effects of web-tapered I-beams, where the deformation compatibilities of the two flanges and web are considered in terms of the basic assumptions of thin-walled members. Subsequently, the total potential for the lateral buckling analysis of web-tapered I-beams is developed, based on the classical variational principle for buckling analysis. The lateral buckling loads of web-tapered cantilevers and simply supported beams of I-sections from the proposed theory are compared with those from the finite element (FE) analyses using two shell element models and two widely used beam element models. The two beam element models respectively represent the equivalent method using prismatic beam elements and the typical tapered beam theory in existing literature. These comparisons show that the results based on the total potential proposed in this paper are more accurate in predicting the lateral buckling loads of web-tapered I-beams than those in existing theories, indicating that the theory proposed in this paper is superior to existing theories. It is also found that the equivalent method using prismatic beam elements may yield unreliable buckling loads of tapered beams.

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1. Introduction

Tapered beams are widely used in modern constructions, mainly due to their structural efficiency. At present, the web-tapered thin-walled I-beam is one of the most popular tapered beams used in practice. The strength of laterally unrestrained thin-walled beams is frequently governed by the lateral buckling (or flexural-torsional buckling) failure, and hence extensive studies were focused on the lateral buckling of thin-walled beams. Most of these studies have been concerned with prismatic beams (e.g. [11,12,14,15]), and only a few investigations are on tapered beams (e.g. [1,2,4–10,18]).

A convenient method to assess the lateral buckling load of a tapered beam is to divide such a beam into several segments and take each of these segments as an equivalent prismatic beam, as adopted by Brown [5] in the finite difference analysis. In this method, the effects of tapering could not be completely taken into account in the expressions of nonlinear strains, which may lead to incorrect lateral buckling loads [18]. Ronagh et al. [9,10] found the errors in lateral buckling loads caused by this method cannot be eliminated merely using fine mesh configuration in the finite element analysis. Note that similar treatments as in [5] are also adopted in the commercial FE software ANSYS on its tapered beam elements [3], implying that the lateral buckling load of tapered beams using tapered elements of ANSYS is not reliable, as illustrated in the later part of this paper.

Considering the effects of tapering in deriving strains, Kitipornchai and Trahair [6] built the equilibrium equations for the lateral buckling analysis of tapered beams, and numerical results were given by means of the finite difference analysis. This method was extended to tapered beams of mono-symmetric sections [7]. While Wekezer [17], Yang and Yau [18], Bradford and Cuk [4], Ronagh et al. [9], and recently Andrade and Camotim [1] investigated the lateral buckling of tapered beams employing the FE method, based on their total potentials presented. It is worth mentioning that in all literature mentioned above strains of each part of tapered beam, i.e. two flanges and web, are obtained using the same relationship between the displacements and strains, although it may be not explicitly stated. For example, the linear longitudinal strain of a point in beam, regardless of its location, is derived by differentiating the linear longitudinal displacement of this point with respect to the longitudinal axis of this beam. This
methodology has been found to violate the basic assumptions of thin-walled members and may result in incorrect results in stress analysis, especially for beams of serious tapering [13]. In this paper, an attempt is made to study the lateral buckling of doubly symmetric web-tapered I-beams. The basic assumptions for thin-walled members, proposed by Vlasov [15], are adopted to derive the strain-displacement relationships for each plate of the tapered beam in order to ensure that the effects of tapering are completely considered in the framework of thin-walled theory. Based on these relationships, new equivalent sectional properties of web-tapered beams are presented. Subsequently, the total potential of web-tapered I-beams, based on the classical variational principle, is presented. Using this total potential, a finite element program is then developed for assessment of lateral buckling load of tapered beams, with which the results of the present theory are compared with those from shell element modelling and typical existing theories.

2. Deformation analysis

For a web-tapered thin-walled beam of I-section shown in Fig. 1, a right-handed coordinate system $x$, $y$ and $z$ is chosen, in which $z$ axis coincides with the centroidal axis and $x$ and $y$ axes coincide with the principal axes of the cross sections. Along the longitudinal direction ($z$ axis), the width and thickness of flanges remain constant, while the height of web varies linearly. The height of section at a distance of $z$ from the small end is then given by

$$h = h_S + (h_L - h_S) \frac{z}{L}$$

(1)

in which $L$ is length of the beam, $h_S$ and $h_L$ are respectively the distances between the centroids of two flanges at the small and large ends (Fig. 1(b)).

Basic deformations, including the axial deformation, bending and torsion, of web-tapered beams of doubly symmetric I-sections are studied in this section prior to the further investigations on web-tapered I-beams. In analysis, the following assumptions are adopted:

1. Material is linearly elastic and homogeneous;
2. Only thin-walled I-beams are concerned;
3. Cross-sections are rigid in their own planes;

4. The linear shear strain on the middle surface of each plate composing thin-walled beams is negligible;
5. Deformation analyses are in the framework of small deformation theory.

2.1. Axial deformation

For a web-tapered I-beam, the centroidal axes of the two flanges (e.g. $s$ axis of the top flange shown in Fig. 2) are not parallel to the centroidal axis of the beam, $z$ axis, so that the deformations of the two flanges and the web of the beam are studied separately.

Fig. 2 shows the axial deformation of a web-tapered beam segment. The original length of the segment along the $z$ axis is $dz$, while the original length of the flange is $d\alpha_0$. Using the geometric relationships shown in Fig. 2, $d\alpha_0$ can be given by:

$$d\alpha_0 = \sqrt{(dz)^2 + \left(\frac{dh}{2}\right)^2}.$$  

(2a)

According to the rigid profile assumption, each cross-section remains a plane in the deformed configurations, so that if this segment has an axial deformation $dw$ (Fig. 2), the length of top flange becomes

$$d\alpha_1 = \sqrt{(dz + dw)^2 + \left(\frac{dh}{2}\right)^2}.$$  

(2b)
The tensile strain of the top flange in the $s$ direction due to the axial deformation of $dw$ is given by

$$
\varepsilon_{f,s} = \frac{ds_1 - ds_0}{ds_0} = \frac{dz}{ds_0} \frac{dw}{ds_0} + \frac{1}{2} \left( \frac{dw}{ds_0} \right)^2.
$$

(3)

Neglecting the nonlinear terms in Eq. (3) and making use of $\frac{dz}{ds_0} = \cos \alpha$, the linear strain of the top flange in the $s$ direction is

$$
\varepsilon_{f,s}^L = \frac{dz}{ds_0} \frac{dw}{ds_0} = \left( \frac{dz}{ds_0} \right)^2 \frac{dw}{dz} = \cos^2 \alpha \frac{dw}{dz}.
$$

(4)

where $\alpha$ is the angle between the $s$ and $z$ axes (Fig. 2), representing the tapering ratio of the tapered beam.

Considering Eq. (4), the axial force of top flange in the $s$ direction is

$$
N_{f,s} = E A_f \varepsilon_{f,s}^L = E A_f \cos^2 \alpha w'.
$$

(5)

where $(\gamma') = \frac{dw}{dz}$.

The component of $N_{f,s}$ in the $z$ direction is given by:

$$
N_f = N_{f,s} \cos \alpha = E A_f w' \cos^3 \alpha
$$

(6)

in which the area of flange is $A_f = t_f b$.

According to conventional beam theory, the strain of the web in the $z$ direction (Fig. 2) is $w'$, and hence the axial force on the web is

$$
N_w = \int_{-h/2}^{h/2} E t_w \varepsilon_w dy = \int_{-h/2}^{h/2} E t_w w' dy = E w' A_w.
$$

(7)

where the area of the web is $A_w = t_u h_w$. Considering Eqs. (6) and (7), the axial force of a web-tapered member can be given by

$$
N = 2 N_f + N_w = E w' \left( 2 A_f \cos^3 \alpha + A_w \right) = E A_w
$$

(8)

in which

$$
A = 2 A_f \cos^3 \alpha + A_w.
$$

(9)

2.2. Bending about $x$ axis

Fig. 3 shows the longitudinal deformation of the web-tapered segment under bending about the $x$ axis. It should be noted that the displacement of this segment in vertical direction, i.e. along the $y$ axis, is not included in this schematic diagram, otherwise, each cross-section of the beam should be perpendicular to the $z$ axis in the deformed configuration.

In terms of the beam bending about the $x$ axis shown in Fig. 3, the length of top flange in the deformed configuration, $s_1$, becomes

$$
ds_1 = \sqrt{\left( \frac{dz}{du} \right)^2 + \left( \frac{du}{du} - dv \right)^2}.
$$

(10)

where $u$ and $v+dv$ are respectively the vertical displacements of the small and large ends of the segment. Considering Eqs. (10), (2) and (4) the axial strain and axial force for the top flange in the $s_1$ direction can be respectively given by

$$
\varepsilon_{f,s} = \frac{ds_1 - ds_0}{ds_0} = \frac{dz}{ds_0} \frac{dw}{ds_0} + \frac{1}{2} \left( \frac{dw}{ds_0} \right)^2.
$$

(11)

$$
N_{f,s} = E A_f \varepsilon_{f,s}^L = E A_f \cos^2 \alpha w'.
$$

(12a)

$$
N_{f,s} = E A_f \frac{h}{2} v'' \cos^2 \alpha.
$$

(12b)

Subsequently, the axial force on bottom flange in the $s_2$ direction is:

$$
N_{f,s2} = -E A_f \frac{h}{2} v'' \cos^2 \alpha.
$$

(13)

The moment of cross-section can be given by

$$
M_x = N_{f,s1} \cos \alpha \frac{h}{2} + N_{f,s2} \cos \alpha \frac{h}{2} + M_w
$$

(14)

where $M_w$ is the contribution of the web and can be obtained by

$$
M_w = \int_{-h/2}^{h/2} \sigma_w t_w y dy = \frac{E}{12} t_u h^3 v''.
$$

(15)

Substitution of Eqs. (12a), (12b) and (14) into Eq. (13) yields

$$
M_x = -EI_x v''
$$

(16)

in which the equivalent second moment of area about the $x$ axis is

$$
I_x = \frac{A_f h^2 \cos^3 \alpha}{2} + \frac{1}{12} t_u h^3.
$$

2.3. Bending about $y$ axis

Since tapering of web-tapered I-beams has no influence on the lateral displacements of the flanges, the moments of two flanges about the weak axis ($y$ axis) can be given by

$$
M_{f,n1} = -EI_y \frac{d^2 u}{ds_1^2} = -EI_y u'' \cos^2 \alpha
$$

(17a)

$$
M_{f,n2} = -EI_y \frac{d^2 u}{ds_2^2} = -EI_y u'' \cos^2 \alpha
$$

(17b)
in which \( I_{yf} = \frac{t_f b_f^3}{12} \); \( M_{f,n1} \) and \( M_{f,n2} \) are the bending moments of the top and bottom flanges about their normal axes, respectively.

The bending moment of cross-section about the \( y \) axis is given by:

\[
M_y = 2M_{f,n1} \cos \alpha = -E \left( 2I_{yf} \cos^3 \alpha \right) u'' = -EI_y u'' \tag{18}
\]

where the equivalent second moment of area about the \( y \) axis is

\[
I_y = \frac{t_f b_f^3 \cos^3 \alpha}{6}. \tag{19}
\]

2.4. Free torsional deformation

As shown in Fig. 4, when a web-tapered I-beam twists about the \( z \) axis with a twisting angle \( \theta \), the twisting angle of top flange about the \( s_1 \) axis is

\[
\theta_{s1} = \theta \cos \alpha. \tag{20}
\]

Consequently, the twist of top flange about the \( s_1 \) axis can be given by

\[
\frac{d\theta_{s1}}{ds_1} = \theta' \cos^2 \alpha. \tag{21}
\]

In terms of the classical thin-walled theory \[15\], the resultant free torsional torque of top flange about the \( s_1 \) axis is given by:

\[
T_{1,s1} = GJ_f \theta' \cos^2 \alpha \tag{22}
\]

where \( J_f = \frac{t_f^3 b_f}{12} \). Considering the free torsional torque of web \( T_w = G\theta' t_w^3 b_w/3 \), the torque of cross-section due to the free torsional deformation about the \( z \) axis is

\[
T_{s1} = 2T_{1,s1} \cos \alpha + T_w = G \left( 2J_f \cos^3 \alpha + \frac{t_w^3 b_w}{3} \right) \theta' = GJ \theta' \tag{23}
\]

in which the equivalent free torsional constant of section is

\[
J = \frac{2t_f^3 b_f}{3} \cos^3 \alpha + \frac{t_w^3 b_w}{3}. \tag{24}
\]

It should be noted that the twisting components of the two flanges about their normal axes result in a part of the warping torque of cross-section (Fig. 4), which will be discussed later.

2.5. Warping deformation

The assumption of zero linear shear strain on the middle-surface of top flange leads to:

\[
y_{xs1} = \frac{\partial w_{s1}}{\partial x} + \frac{\partial \bar{u}}{\partial s_1} = 0 \tag{25}
\]

in which \( w_{s1} \) and \( \bar{u} \) are the displacements of an arbitrary point on the middle-surface of top flange along the \( s_1 \) and \( x \) axes, respectively. Due to the rigid profile assumption, the lateral displacement of top flange is \( \bar{u} = h\theta/2 \). Substituting this relationship into Eq. (25) yields

\[
\frac{\partial w_{s1}}{\partial x} = -\cos \alpha \frac{\partial \bar{u}}{\partial z} = -\left( h\theta/2 + \tan \alpha \theta \right) \cos \alpha \tag{26}
\]

Integration of Eq. (26) leads to

\[
w_{s1} = -x \cos \alpha \left( h\theta'/2 + \tan \alpha \theta \right). \tag{27}
\]

It should be noted that the second term in Eq. (27), \( -x \tan \alpha \cos \alpha \theta \), is the contribution of the top flange bending about the \( y \) axis. The warping strain of top flange in the \( s_1 \) direction can be obtained using Eq. (27) as:

\[
\varepsilon_{s1} = \frac{dw_{s1}}{ds_1} = -x \left( h\theta''/2 + 2 \tan \alpha \theta \right) \cos^2 \alpha. \tag{28}
\]

The bending moment of the top flange about the \( n_1 \) axis is then given by:

\[
M_{y,f1} = \int_{-b_f/2}^{b_f/2} \sigma xdA_f = -EI_{yf} \cos^2 \alpha \left( h\theta''/2 + 2 \tan \alpha \theta \right). \tag{29a}
\]

Consequently, the shear force in top flange along the \( x \) axis due to the warping deformation (restrained torsion) is

\[
Q_{axs,f1} = \frac{dM_{y,f1}}{ds_1} = -EI_{yf} \left( h\theta''/2 + 3 \theta'' \tan \alpha \right) \cos^3 \alpha. \tag{29b}
\]

where the positive direction of \( Q_{axs,f1} \) is shown in Fig. 6(b).

Following the same process as the top flange, for the bottom flange:

\[
w_{s2} = x \left( h\theta'/2 + \tan \alpha \theta \right) \cos \alpha \tag{30a}
\]

\[
\varepsilon_{s2} = \frac{dw_{s2}}{ds_2} = x \left( h\theta''/2 + 2 \tan \alpha \theta' \right) \cos^2 \alpha \tag{30b}
\]

\[
M_{y,f2} = EI_{yf} \cos^2 \alpha \left( h\theta''/2 + 2 \tan \alpha \theta' \right) \tag{30c}
\]

\[
Q_{axs,f2} = \frac{dM_{y,f2}}{ds_2} = EI_{yf} \left( h\theta''/2 + 3 \theta'' \tan \alpha \right) \cos^3 \alpha. \tag{30d}
\]
Making use of Eqs. (29a) and (30c), the bimoment of cross-section is given by:

\[ B_\alpha = \frac{h}{2} M_{x,f1} \cos \alpha - \frac{h}{2} M_{y,f2} \cos \alpha \]
\[ = -\frac{h}{2} E I_{1f} \cos^3 \alpha (h \theta'' / 2 + 2 \tan \alpha \theta') \]
\[ = -E(I_{w} \theta'')' \]  
(31)

in which the equivalent warping constant of cross-sections is

\[ I_{w} = 2 I_{w1} \cos^3 \alpha \left( \frac{h}{2} \right) = \frac{h^2 I_y}{4} \]  
(32)

where \( I_y \) is defined in Eq. (19).

The shear forces in two flanges due to the warping deformation (Eqs. (29b) and (30d)) do not result in a shear force but a part of warping torque of cross-section:

\[ M_{w1} = Q_{ox,f1} \frac{h}{2} - Q_{ox,f2} \frac{h}{2} = -3 E I_{w1} \theta'' - E I_{w1} \theta'''. \]  
(33a)

The second part of warping torsion is the components of \( M_{y,f1} \) and \( M_{y,f2} \) in \( z \) direction shown in Fig. 5(a):

\[ M_{w2} = -M_{y,f1} \sin \alpha + M_{y,f2} \sin \alpha \]
\[ = 2 E I_{w0} \theta'' + E I_{w0} \theta''. \]  
(33b)

Then, the warping torque of cross-section can be given by

\[ M_{w} = M_{w0} + M_{w2} = -E(I_{w} \theta''') + E I_{w0} \theta'. \]  
(33c)

It is worth mentioning that in Eq. (33c) the relationship between the bimoment and warping torque for prismatic thin-walled member, \( M_{w} = B'_{w0} \), is no longer applicable to the web-tapered I-beam.

3. Prebuckling stresses

In order to develop the total potential for lateral buckling analysis of web-tapered I-beams, the prebuckling stresses, including the longitudinal normal stress, shear stress and transverse normal stress, in each plate due to in-plane bending are investigated in this section.

3.1. Longitudinal normal stress

Making use of the Hooke’s law and considering Eq. (11) yield

\[ \sigma_{f,x1} = \frac{E h \cos^2 \alpha}{2} v'' \]  
(34a)

\[ \sigma_{f,x2} = -\frac{E h \cos^2 \alpha}{2} v'' \]  
(34b)

\[ \sigma_{w,z} = -E y v'' \]  
(34c)

where \( \sigma_{f,x1}, \sigma_{f,x2} \) and \( \sigma_{w,z} \) are the longitudinal normal stress of the top flange in the \( s_1 \) direction, the bottom flange in the \( s_2 \) direction and the web in \( z \) direction, respectively.
Substituting Eqs. (35a) and (35b) into Eqs. (36a) and (36b) yields

\[
\sigma_{s1} = \sigma_{w1} \tan^2 \alpha + 2\tau_{ns1} \tan \alpha + \frac{\cos 2\alpha}{\cos^2 \alpha} \sigma_n
\]  \tag{37a}

\[
\tau_{w1} = -\sigma_{w1} \tan \alpha - \tau_{fs1} + \sigma_n \tan \alpha.
\]  \tag{37b}

Eqs. (37a) and (37b) can be rewritten as:

\[
\sigma_{s1} = \sigma_{w1} \tan^2 \alpha + 2\tau_{ns1} \tan \alpha - \frac{q}{t_w}
\]  \tag{38a}

\[
\tau_{w1} = -\sigma_{w1} \tan \alpha - \bar{\tau}_{ns1}
\]  \tag{38b}

in which:

\[
\bar{\tau}_{ns1} t_w = EA f \cos^3 \alpha \left( \frac{h}{2} v'' \right)
\]  \tag{39a}

\[
\sigma_{w1} = E \frac{h}{2} v''.
\]  \tag{39b}

The equilibrium condition of a web element in the longitudinal direction \( z \) leads to

\[
\frac{\partial \sigma_{w}}{\partial z} + \frac{\partial \tau_{w}}{\partial y} = 0.
\]  \tag{40}

Substituting \( \sigma_w = -Eyv'' \) into Eq. (40) and making use of the boundary condition \( \tau_{w}|_{-h/2} = \tau_{w1} \), the shear stress of the web \( \tau_{w} \) can be given by

\[
\tau_{w1} t_w = -Ev'' \left( A f \cos^3 \alpha + \frac{A_w}{2} \right) \tan \alpha + ES_x v'''
\]

\[
= E \left( S_x v''' \right),
\]  \tag{41}

where \( S_x = -\frac{1}{3} A f / h \cos^3 \alpha + \frac{1}{12} t_w \left( y^2 - h^2 / 4 \right) \) and \( A_w = t_w h \). The shear force of the cross-section can be derived by integrating \( \tau_{w} \) along the height of web:

\[
Q_{yw} = \int_{-h/2}^{h/2} \tau_{w1} t_w dy
\]

\[
= -Ev'' \left( A f \cos^3 \alpha + \frac{A_w}{2} \right) h \tan \alpha - EI_x v'''
\]

in deriving Eq. (42) the integral relationship \( \int_{-h/2}^{h/2} S_x dy = -I_x \) has been used. In the beam bending about \( x \) axis, the shear force of cross-section is from the contribution of the web, i.e.

\[
Q_y = Q_{yw} = -E \left( I_x v''' \right) = \frac{dM_x}{dz}.
\]  \tag{43}

The force equilibrium of a top flange element in the \( s_1 \) direction leads to:

\[
\frac{\partial \sigma_{fs1}}{\partial s_1} + \frac{\partial \tau_{x1}}{\partial x} = 0
\]  \tag{44}

in which \( \tau_{x1} \) is the shear stress in the top flange in the \( x-s_1 \) plane. Substitution of Eq. (34a) into Eq. (43) and considering the zero shear stress condition at the free edge yield

\[
x < 0: \quad \tau_{x1} = E \cos^3 \alpha \left( x + \frac{b_f}{2} \right) \left( h_1 v''' \right)
\]  \tag{45a}

\[
x > 0: \quad \tau_{x1} = E \cos^3 \alpha \left( x - \frac{b_f}{2} \right) \left( h_1 v''' \right).
\]  \tag{45b}

The shear stress in bottom flange can also be obtained:

\[
x < 0: \quad \tau_{x2} = -E \cos^3 \alpha \left( x + \frac{b_f}{2} \right) \left( h_1 v''' \right)
\]  \tag{46a}

\[
x > 0: \quad \tau_{x2} = E \cos^3 \alpha \left( \frac{b_f}{2} - x \right) \left( h_1 v''' \right).
\]  \tag{46b}

Note that the sign of shear stresses in Eqs. (45) and (46) is the same as shown in Fig. 5(b).

### 3.3. Transverse normal stress

The transverse normal stress in thin-walled beams has been normally assumed to be negligible in classical theories, while Tong and Zhang [12] pointed out that this type of stress is essential in retaining the force equilibrium of plate elements in their transverse directions and is of importance in developing the total potential for lateral buckling analysis of these beams.

Considering the force equilibrium of a top flange element in the \( x \) direction yields:

\[
\frac{\partial \sigma_{x1}}{\partial x} + \frac{\partial \tau_{x1}}{\partial s_1} = 0.
\]  \tag{47}

Substituting Eqs. (45a) and (45b) into Eq. (47), the transverse normal stress in top flange can be derived:

\[
\sigma_{s1} = \frac{1}{4} \cos^4 \alpha \left( hv''' \right) \left( \frac{b_f}{2} - |x| \right)^2.
\]  \tag{48a}

Then, the transverse normal stress in bottom flange can be given by

\[
\sigma_{s2} = \frac{1}{4} \cos^4 \alpha \left( hv''' \right) \left( \frac{b_f}{2} - |x| \right)^2.
\]  \tag{48b}
The derivation of transverse stress in web is based on the vertical equilibrium of web element:
\[
\frac{\partial \sigma_{yw}}{\partial y} + \frac{\partial \tau_{iw}}{\partial z} = 0. \tag{49}
\]
Substituting Eq. (41) into Eq. (49) and considering the boundary condition
\[
\sigma_{yw}\big|_{y=-h/2} = \sigma_1 \tag{50a}
\]
the transverse stress in web (in the vertical direction) can be given by
\[
\sigma_{yw}t_w = -q - \left( \frac{M_x D_{x1}}{I_x} \right)'' \tag{50b}
\]
in which
\[
D_{x1} = \left( \frac{h A_f}{2} \cos^3 \alpha + \frac{t_w h^2}{8} \right) (y + \frac{h}{2}) - \frac{1}{6} t_w \left( y^3 + \frac{h^3}{8} \right). \tag{50c}
\]
It should be noted that the boundary condition in Eq. (50a) depends on the loading position of \( q \). For example, when the uniformly distributed load \( q \) is applied at the bottom flange, this condition becomes:
\[
\sigma_{yw}\big|_{y=h/2} = \sigma_2. \tag{51a}
\]
In this case, the transverse stress in the web can be expressed by
\[
\sigma_{yw}t_w = q - \left( \frac{M_x D_{x2}}{I_x} \right)'' \tag{51b}
\]
in which
\[
D_{x2} = \left( \frac{h A_f}{2} \cos^3 \alpha + \frac{t_w h^2}{8} \right) (y - \frac{h}{2}) - \frac{1}{6} t_w \left( y^3 - \frac{h^3}{8} \right). \tag{51c}
\]

4. Total potential for lateral buckling analysis of web-tapered I-beams

A careful study carried out by Tong and Zhang [11] revealed that the total potential for lateral buckling analysis of thin-walled beams should be derived on the basis of the variational principle for buckling analysis, with which a new and more rational total potential for prismatic thin-walled beams has been developed in [12,19]. In this section, this variational principle will be employed to develop the total potential for web-tapered I-beams. For ease of description, a brief introduction on the variational principle is first made below.

4.1. Variational principle for buckling analysis

Buckling of a thin walled member may be treated as an initial stress problem of variational principle, with which the prebuckling stresses are treated as the initial stresses and the buckling displacements as the virtual displacements. After rigorous development in [16], the variational principle for the buckling analysis of an initial stress problem is given by:
\[
\iint_V \left( \sigma_{ij} \delta \varepsilon_{ij} + k \sigma_0^0 u_{k,i} \delta u_{k,i} \right) dV = 0 \tag{52}
\]
where \( \sigma_{ij} \) (\( i, j = 1, 2, 3 \)) are the initial stress components; \( u_{k,i} \) and \( u_{k,j} \) (\( i, j = 1, 2, 3 \)) are the displacement components of buckling; \( \varepsilon_{ij} \) and \( \sigma_{ij} \) (\( i, j = 1, 2, 3 \)) are the linear stress and strain components of buckling; \( k \) is the load factor. The total potential corresponding to Eq. (52) can be gained as:
\[
\Pi = \iiint_V \left[ \frac{1}{2} \left( \sigma_{ij} \varepsilon_{ij} + k \sigma_0^0 u_{k,i} u_{k,j} \right) \right] dV. \tag{53}
\]
The first term in Eq. (53) is the linear strain energy and the second one is the nonlinear strain energy. It should be noted that the load potential widely considered in the total potential of existing theories, being the work done by external load on the nonlinear components of buckling displacements, does not appear in Eq. (53).

4.2. Nonlinear strains

When the lateral buckling occurs, the thin-walled beams experience a sudden flexural displacement \( u \) and torsion \( \theta \). The displacements of an arbitrary point on middle surface of top flange \( \bar{w}_{s1}, \bar{w}_{n1} \) respectively in the \( s_1, x \) and \( n_1 \) directions can be expressed using \( u \) and \( \theta \), according to the basic assumptions of thin-walled members as:
\[
\bar{w}_{s1} = -x \cos \alpha \left( u' + \theta' \frac{h}{2} \right) - x \theta' \sin \alpha \tag{54a}
\]
\[
\bar{w}_{n1} = -x \sin \alpha \left( u' + \frac{\theta h'}{2} \right) + x \theta \cos \alpha \tag{54b}
\]
\[
\bar{u} = u + \frac{h \theta}{2}. \tag{54c}
\]
Making use of the displacements in Eqs. (54a)–(54c), the nonlinear longitudinal normal strain in the top flange in the \( s_1 \) direction can be obtained:
\[
\varepsilon_{s1}^N = \frac{1}{2} \left[ \left( \frac{\partial \bar{w}_{s1}}{\partial s_1} \right)^2 + \left( \frac{\partial \bar{w}_{n1}}{\partial s_1} \right)^2 + \left( \frac{\partial \bar{v}_{n1}}{\partial s_1} \right)^2 \right]
\]
\[
= \frac{1}{2} \cos^2 \alpha \left[ x^2 \left( u'' + \frac{h \theta''}{2} + \theta' \tan \alpha \right)^2 + (x \theta')^2 \right]
\]
\[
+ \left( u' + \frac{h \theta'}{2} + \theta \tan \alpha \right)^2. \tag{55}
\]
Neglecting the higher-order terms in Eq. (55) yields
\[
\varepsilon_{s1}^N = \frac{1}{2} \left( x \theta' \right)^2 + \frac{1}{2} \left( u' \cos^2 \alpha + \frac{h \theta'}{2} + \theta \tan \alpha \right)^2. \tag{56a}
\]
Using the displacements in Eqs. (54a)–(54c) and also neglecting the higher-order terms, the nonlinear shear and
transverse normal strains in the top flange can be given by:

\[
y_{s,1} = \frac{\partial \tilde{v}_{s,1}}{\partial x} \frac{\partial \tilde{v}_{s,1}}{\partial s} + \frac{\partial \tilde{u}}{\partial x} \frac{\partial \tilde{u}}{\partial s} + \frac{\partial \tilde{w}_{s,1}}{\partial x} \frac{\partial \tilde{w}_{s,1}}{\partial s} = x \theta' \cos \alpha \\
\epsilon_{s,1}^N = \frac{1}{2} \left[ \left( \frac{\partial \tilde{v}_{s,1}}{\partial x} \right)^2 + \left( \frac{\partial \tilde{u}}{\partial x} \right)^2 + \left( \frac{\partial \tilde{v}_{s,1}}{\partial x} \right)^2 \right] = \frac{1}{2} \dot{\theta}^2. 
\]

(56b)

For the bottom flange, the nonlinear strains can be easily given by

\[
\begin{align*}
\epsilon_{n,2}^N &= \frac{1}{2} \left( x \theta' \right)^2 + \frac{1}{2} \left( u' - h \theta' - \theta \tan \alpha \right)^2 \cos^2 \alpha \\
\gamma_{n,2}^N &= \frac{1}{2} \theta^2. 
\end{align*}
\]

(57a)

(57b)

(57c)

The nonlinear strains in the web are the same as those in prismatic beams:

\[
\begin{align*}
\epsilon_{w,2}^N &= \frac{1}{2} \left( u'^2 + y^2 \theta'^2 - 2yu' \theta' \right) \\
\gamma_{yz}^N &= -\theta \left( u' - \gamma' \right) \\
\dot{\epsilon}_{sw}^N &= \frac{1}{2} \theta^2.
\end{align*}
\]

(58a)

(58b)

(58c)

4.3. Linear strain energy

The linear strain energy of web-tapered I-beams due to the beam bending about the y axis is

\[
U_u = \frac{1}{2} \int_0^L \int_A \sigma_{f,s,1}^L \varepsilon_{f,s,1}^N dA \, ds_1 = \frac{1}{2} \int_0^L \int_{A_f} \sigma_{f,s,1}^L \varepsilon_{f,s,1}^N dA_f d s_1.
\]

(59a)

in which the longitudinal stress and strain are induced by the bending about y axis. Utilizing \( \sigma_{f,s,1} = E \varepsilon_{f,s,1}^L \) and \( ds_1 = ds_1 \cos \alpha \), Eq. (59a) can be rewritten as

\[
U_u = \frac{1}{2} \int_0^L \left( E I_s u''^2 \right) \, dz.
\]

(59b)

The strain energy due to the free torsion of beam is given by

\[
U_{st} = \frac{1}{2} \int_0^L \left( 2T_f \theta' \cos \alpha \right) \, ds_1 + \frac{1}{2} \int_0^L \left( T_w \theta' \right) \, dz = \frac{1}{2} \int_0^L \left( 2T_f \theta' + T_w \theta' \right) \, dz = \frac{1}{2} \int_0^L \left( GJ \theta'^2 \right) \, dz.
\]

(59c)

Another part of the linear strain energy is the one corresponding to the restrained torsion of beam. Considering Eqs. (59a) and (28), this strain energy is given by

\[
U_{st} = \frac{1}{2} \int_0^L \left( E \left( I_S \theta'^2 + 2l_{10} \theta'^2 + 2l_{10} \theta'' \theta' \right) \right) \, dz.
\]

(59d)

Combining Eqs. (59a)–(59d), the linear strain energy of web-tapered I-beam in lateral buckling is

\[
U_L = \frac{1}{2} \int_0^L \left[ E I_s u''^2 + (GJ + 2EI_w') \theta'^2 + EI_{1w} \theta'^2 + 2EI_{1w} \theta'' \theta' \right] \, dz.
\]

(60)

4.4. Nonlinear strain energy

The nonlinear strain energy involved in lateral buckling analysis includes the nonlinear longitudinal normal strain energy, nonlinear shear strain energy and nonlinear transverse normal strain energy [12]. In the following, these three parts of the nonlinear strain energy will be derived respectively.

The nonlinear longitudinal normal strain energy consists of contributions of the two flanges and the web. Using the linear stresses and nonlinear strains in the two flanges and the web developed in Sections 3.1 and 4.2 yields

\[
U_{\sigma}^N = \int_0^L \int_A \sigma_{f,s,1}^N \varepsilon_{f,s,1}^N dA_1 \, ds_1 + \int_0^L \int_{A_2} \sigma_{w,c,z}^N \varepsilon_{w,c,z}^N dA_w \, dz
\]

\[
= \frac{1}{2} \int_0^L \int_A \left\{ \frac{E v'' h \cos^2 \alpha}{2} \left[ \left( x \theta' \right)^2 + \left( u' \cos^2 \alpha \right) + \frac{h \theta'}{2} + \theta \tan \alpha \right] \right\} \, dA_1 \, ds_1
\]

\[
+ \frac{1}{2} \int_0^L \int_A \left\{ -\frac{E v'' h \cos^2 \alpha}{2} \left[ \left( x \theta' \right)^2 + \left( u' \cos^2 \alpha \right) - \frac{h \theta'}{2} - \theta \tan \alpha \right] \right\} \, dA_2 \, ds_2
\]

\[
+ \frac{1}{2} \int_0^L \int_A \left\{ [ -E v'' y \left( u'^2 + y^2 \theta'^2 - 2yu' \theta' \right) ] \right\} \, dA_w \, dz
\]

\[
= -\frac{1}{2} \int_0^L \left[ 2^\frac{1}{2} \left( \frac{M_x}{I_x} \right) \left( I_s u' \theta' \right) + h A_f \cos^3 \alpha \tan \alpha u' \theta' \right] \, dz.
\]

(61)

The nonlinear shear strain energy is

\[
U_{\tau} = \int_0^L \int_A \tau_{x=x_1} \gamma_{x=x_1}^N dA_1 \, ds_1 + \int_0^L \int_{A_2} \tau_{x=x_2} \gamma_{x=x_2}^N dA_1 d s_2
\]

\[
+ \int_0^L \int_A \tau_{y-c} \gamma_{y-c}^N dA_w \, dz
\]

\[
= \int_0^L \int_{-h/2}^{h/2} \left[ \left( \frac{M_x}{I_x} \right)' (u' \theta' - y \theta' \theta) \right] \, dy dz
\]

\[
= -\frac{1}{2} \int_0^L \left[ 2 \left( \left( \frac{M_x}{I_x} \right)' I_x - \frac{M_x}{I_x} S_S' h \right) u' \theta' \right] \, dz.
\]

(62)
In the derivation of Eq. (62), the following integral relationships have been used:

\[
\int_{-h/2}^{h/2} S_x dy = -I_x, \quad (63a)
\]

\[
\int_{-h/2}^{h/2} S_x y dy = 0, \quad (63b)
\]

\[
\int_{-h/2}^{h/2} S'_x dy = S'h, \quad (63c)
\]

\[
\int_{-h/2}^{h/2} S'_x y dy = 0. \quad (63d)
\]

The nonlinear transverse normal strain energy is

\[
U_{\sigma N} = \int_0^L \int_{A_1} \sigma_x^l e_{x1}^N dA_1 d\alpha_1 + \int_0^L \int_{A_2} \sigma_x^l e_{x2}^N dA_2 d\alpha_2 + \int_0^L \int_{A_3} \sigma_y^l e_{y}^N dA_3 d\alpha_3
\]

\[
= -\frac{1}{2} \int_0^L \left\{ q h + \frac{h}{2} \left[ \left( \frac{M_x}{l_x} \right) I''_x + 2 \left( \frac{M_x}{l_x} \right)' I'_x \right] \right\} \theta^2 d\alpha_3
\]

\[
= -\frac{1}{2} \int_0^L \left\{ q h + \frac{h}{2} \left[ \left( \frac{M_x}{l_x} \right) \right]'' I_x \right\} \theta^2 d\alpha_3
\]

\[
= -\frac{1}{2} \int_0^L \left\{ q h + \frac{h}{2} M''_x \right\} \theta^2 d\alpha_3. \quad (64)
\]

In deriving Eq. (64), the following relationships have been used

\[
\int_{-h/2}^{h/2} D_{x1} dy = \frac{I_x h}{2}, \quad (65a)
\]

\[
\int_{-h/2}^{h/2} D_{x2} dy = \frac{I'_x h}{2}, \quad (65b)
\]

\[
\int_{-h/2}^{h/2} D_{x3} dy = \frac{I''_x h}{2}. \quad (65c)
\]

Considering \( M''_x = -q \), Eq. (64) can be rewritten as

\[
U_{\sigma N} = -\frac{1}{2} \int_0^L \left( \frac{h}{2} \theta^2 \right) d\alpha_3. \quad (66)
\]

In more general cases, if the uniformly distributed load \( q \) is applied at a distance of \( a_q \) above the shear centre of cross-sections, following the similar derivations above, Eq. (64) can be rewritten as

\[
U_{\sigma N} = -\frac{1}{2} \int_0^L \left( q a_q \theta^2 \right) d\alpha_3. \quad (67)
\]

Considering \( I_x = -S'h + A \cos^3 \alpha \tan \alpha h \), the total potential for lateral buckling analysis of web-tapered I-beams is

\[
\Pi = U_L + U_N + U'_N + U_{\sigma N}
\]

\[
= \frac{1}{2} \int_0^L \left[ E I_x u''^2 + (G J + 2 EI'_w) \theta'^2 + EI_\theta \theta'^2 \right.\]

\[+ 2EI'_w \theta' \theta'' \left.\right] d\alpha_3
\]

\[-\frac{1}{2} \int_0^L \left( 2 (M_x \theta)' u' + qa_q \theta^2 \right) d\alpha_3. \quad (68)
\]

According to Tong and Zhang [11,12], in the case of a web-tapered I-beams subjected to the uniformly distributed load of \( q \) and a concentrated load of \( P \), the total potential in Eq. (68) is:

\[
\Pi = \frac{1}{2} \int_0^L \left[ E I_x u''^2 + (G J + 2 EI'_w) \theta'^2 + EI_\theta \theta'^2 \right.\]

\[+ 2EI'_w \theta' \theta'' \left.\right] d\alpha_3 \]

\[-\frac{1}{2} \int_0^L \left( 2 (M_x \theta)' u' + qa_q \theta^2 \right) d\alpha_3 \]

\[-\frac{1}{2} Pa_P \theta^2. \quad (69)
\]

in which \( a_P \) is the distance of loading point of \( P \) above the shear centre of cross-section, and \( \theta_z \) is the rotation of cross-section where \( P \) applied.

It is worth mentioning that as far as the web-tapered beams of doubly symmetric I-section are concerned, the total potentials proposed by Yang and Yau [18], Bradford and Cuk [4], Andrade and Camotim [1] have the same expression as in Eq. (69). However, the sectional properties of tapered beams used in Eq. (69) \( (I_y, I_w, \text{and } J) \) are derived by considering the effects of tapering and are different with those used in the other studies, being the same with those for prismatic beams.

5. Results and discussions

The total potential proposed in this paper (Eq. (69)) is verified by comparing the lateral buckling loads of web-tapered I-beams with results from typical theories in existing studies and FE analyses using shell element and beam element models of ANSYS. In the following comparisons, some of the examples in [2] were used.

5.1. FE modelling

The eigenvalue method is adopted in calculating the lateral buckling load of web-tapered I-beams. In development of FE program based on the total potential presented in Eq. (69), the linear and geometric stiffness matrices are easily obtained following the widely used procedure [4,18]. This method is also employed to obtain lateral buckling loads of total potentials in existing literature.

Essentially, a thin-walled member is composed of thin plates, and hence the buckling load of thin-walled beam can also be calculated from shell element modelling in FE analysis [2,20]. Generally speaking, the results from shell element modelling are more precise than those from beam element modelling, as less assumptions are adopted in shell theory than beam theory. From this point of view, results from shell element modelling can be treated as reference results in verifying beam theories. However, as far as the lateral buckling is concerned, the buckling loads from shell element modelling may not be reliable in some cases, particularly when the beam is short [2],
because the local deformations may influence the buckling mode with shell element modelling [2,20]. One approach to prevent local deformations in shell element modelling is to place stiffeners along the beam span, with which the “rigid section” assumption of thin-walled beam can be satisfied. While, on the other hand, these stiffeners may also provide restraints to the warping deformations of the stiffened cross-sections, which therefore leads to overestimation of lateral buckling load [20]. In order to avoid this warping restraint, in this study, a new shell element model (Fig. 7(a)) is proposed to investigate the lateral buckling of thin-walled beams, where the beam is modelled using thin shell elements while the stiffeners are modelled using membrane elements (Fig. 7(b)). In such shell element modellings, the stiffened sections can reach a rigid profile without any additional stiffnesses, including flexural stiffness and free and restrained torsional stiffnesses of the beam. Based on a large number of calculations in [20] and this study, it is found that a shell element model with 5–8 such membrane stiffeners along the beam span normally perform very well in predicting the lateral buckling loads of a thin-walled beam.

The commercial FE software ANSYS is adopted in FE modelling using shell elements (Fig. 8). In shell element modelling, the beam is modelled using the elastic thin shell element SHELL63, and with the stiffeners using thin membrane element SHELL41. In some cases, results using the thin walled beam elements BEAM189 of ANSYS, which is capable of modelling tapered beams [3], are also included in the comparisons below.

The web-tapered cantilevers and simply supported beams of I-section are studied as examples in this section. In shell element modelling, the fixed end of the cantilevers is achieved by restraining all freedoms of every node of this section. While in modelling the simply supported beams, the translation displacements in x and y directions of all nodes of support sections are restrained, in addition, the rigid body motion of beam in z direction is prevented by fixing the z displacement of an arbitrary node. These treatments in shell element modelling can satisfy the simply supported condition, as well as the rigid profile requirement of support sections.

5.2. Lateral buckling of prismatic cantilevers

An extreme case of the web-tapered beam, a cantilever of uniform section, is first analyzed to verify the validity of FE modelling presented in this paper, since the lateral buckling loads of prismatic thin-walled beams have been extensively investigated and widely accepted. As shown in Fig. 8, the parameters of cantilevers are $E = 210$ GPa, $v = 0.3$, $h = 600$ mm, $t_f = 10$ mm, $t_w = 8$ mm and $b = 180$ mm, while the length of cantilever $L$ varies from 2.0 to 10.0 m. The lateral buckling loads based on the total potential of Andrade and Camotim [1] are included as the results from the typical theory in existing literature [1,4,18].

Comparisons of lateral buckling loads of cantilevers are made in Table 1 and Fig. 9. In Table 1 and Fig. 9, $Q_{cr}^{S,Andrade}$ is the buckling load from FE analysis using shell element models without any stiffeners conducted by Andrade et al. [2], which is almost identical to $Q_{cr}^{AS}$, from the same FEA modelling in this study, indicating that $Q_{cr}^{AS}$ can be used to represent the results from this type of FE modelling instead of $Q_{cr}^{S,Andrade}$ for additional results in following comparisons. Very good agreements between $Q_{cr}^{Andrade}$, $Q_{cr}$ and $Q_{cr}^{AB}$ in Table 1, respectively being the results based on the beam theories of Andrade and Camotim [1], this paper, and from FE analysis using BEAM 189 of ANSYS, imply that for prismatic beams these three methods can generate the same lateral buckling loads of doubly symmetric thin-walled beams.

It can be seen from Table 1 that $Q_{cr}^{AS}$ (or $Q_{cr}^{S,Andrade}$) is always smaller than those based on beam theory, $Q_{cr}$ (or $Q_{cr}^{Andrade}$), and the differences between these two results are
even more significant for short beams, as also pointed out by Andrade et al. [2]. The results from FE analyses using shell element models with membrane stiffeners, $Q_{cr}^{AS}$, match very well with $Q_{cr}$ (or $Q_{cr}^{Andrade}$), even for short beams. For example, the discrepancy between $Q_{cr}^{AS}$ and $Q_{cr}$ of the beam with $L = 2$ m is only 0.42%, while that between $Q_{cr}^{AS}$ and $Q_{cr}$ reaches 55.74%. Very good agreement between $Q_{cr}^{AS}$ and $Q_{cr}$, even for short beams, imply that $Q_{cr}^{AS}$ is superior to $Q_{cr}^{AS}$ in predicting the lateral buckling loads of thin-walled beams and can be used as the reference results of lateral buckling loads of web-tapered I-beams.

5.3. Lateral buckling of web-tapered beams

Based on the discussions above, the lateral buckling loads from FE analysis using shell element models with membrane stiffeners placed along the beam span, referred to as shell element modelling in the rest part of this paper, are taken as the reference results in verifying the results from the beam theory of this paper.

(a) Lateral buckling of web-tapered beams subjected to concentrated load at the top flange of free end

A web-tapered cantilever of doubly symmetric I-sections, as shown in Fig. 10, is considered. Along the longitudinal axis, the cantilevers have the same dimensions of flanges, while the height of web varies linearly. The centroidal axis of the cantilever, $z$ axis, is perpendicular to its end sections. The material properties are $E = 210$ GPa and $v = 0.3$; while the maximum height of cross-section at the fixed end is $h_L = 600$ mm and the width of flanges is 180 mm; the thicknesses of the flanges and the web are $t_f = 10$ mm and $t_w = 8$ mm, respectively. The span of the cantilevers $L$ and the ratio of $h_S$ to $h_L$, $\beta$, respectively vary from 2.0 to 8.0 m and 0.1 to 1.0. It should be noted that the concentrated load ($Q$) is always applied at the top flange of the free end section.

Fig. 11(a)–(d) show the lateral buckling loads of web-tapered cantilevers (Fig. 10) from different methods. In Fig. 11(a)–(d), $Q_{cr}^{AS}$ are the results from FE analyses using shell element models with 6 stiffeners (2.5 mm in thickness) placed along the longitudinal axis of the cantilevers.

Fig. 11(a)–(d) show that agreements between the lateral buckling loads from different methods strongly depend on the tapering ratio of beams. The tapering ratio of tapered beams relies on the span of beam and the parameter $\beta$, and can be represented by the angle $\alpha$ between the centroidal axes of the beam and flanges. A beam with a short span and a small value of $\beta$ or with a big value of $\alpha$ means having a large tapering ratio. It can be seen from Fig. 11(a)–(d) that $Q_{cr}$ are in very good agreements with $Q_{cr}^{AS}$ for most of the considered beams. However, for very short beams of extreme tapering, $Q_{cr}^{AS}$ is much smaller than results using beam elements (e.g. cantilevers with $L = 2$ m and $\beta < 0.2$ in Fig. 11(a)). This is because obvious local buckling deformations take place in the shell element modelling of these beams. Comparisons between $Q_{cr}^{Andrade}$ and $Q_{cr}$ show that the discrepancies between these two results are significant for beams with large tapering ratios (Fig. 11(a) and (b)). For these beams, the differences between $Q_{cr}^{Andrade}$ and $Q_{cr}$ are also significant because the parameter $\alpha$ may strongly affect the sectional properties of such tapered beams in Eqs. (19), (24) and (32). For the same reasons, $Q_{cr}^{Andrade}$, $Q_{cr}$, and $Q_{cr}^{AS}$ match well with each other for beams with small tapering ratios (Fig. 11(a)–(d)). Note that compared to $Q_{cr}^{AS}$ and $Q_{cr}$, $Q_{cr}^{Andrade}$ always are in some degree of overestimation of the lateral buckling loads of tapered beams.
As illustrated in Fig. 11(a)–(d), $Q_{cr}^{AB}$, representing the results using prismatic beam elements, shows significant differences with the other results, from either tapered beam element modelings or shell element modelings. These differences are notable even for long beams, as shown in Fig. 11(c) and (d). It should be noted that the mesh configurations in calculating results in Fig. 11(a)–(d) are based on convergence studies, which also indicates that, as pointed out by Ronagh et al. [10], the errors of $Q_{cr}^{AB}$ cannot be eliminated by using fine mesh in FE analyses.

(b) Lateral buckling of web-tapered beams subjected to a concentrated load at the shear centre of free end

The second example is the web-tapered cantilevers with a concentrated load at the shear centre of free end (Fig. 12). The parameters of cantilevers are the same as those used in Fig. 10, with the only difference being the loading position along the vertical axis. In calculating $Q_{cr}^{AS}$, similar models as in Fig. 11 are adopted. Due to the discussions above, obvious local deformations may occur in shell element modeling when the length of cantilever is very short. In these cases, the results from shell element modeling do not represent the lateral buckling load of thin-walled beams. On the other hand, the effects of tapering seem very slight for long cantilevers, which may lead to negligible differences in lateral buckling loads from different methods. Consequently, cantilevers of “medium length” are chosen in the comparisons in Fig. 13(a) and (b).

The lateral buckling loads of cantilevers with the length of 3m and 4.5m are compared in Fig. 13(a) and (b), respectively. In these comparisons, advantages of $Q_{cr}$ in predicting the lateral buckling loads of such cantilevers over either $Q_{cr}^{Andrade}$ or $Q_{cr}^{AB}$ can also be exploited. $Q_{cr}^{Andrade}$ shows a little overestimation of lateral buckling loads compared to $Q_{cr}$, especially for beams of serious tapering, the differences between being significant. It should be noted that in Fig. 13(a) and (b), $Q_{cr}^{AS}$ is slightly smaller that $Q_{cr}$ even for prismatic beams, which is due to the local deformation effects in the vicinity of the loading point in web.
It can be seen from the comparisons in Figs. 11(a)–(d) and 13(a) and (b) that \( Q^{AB}_{cr} \) increases monotonically with the parameter \( \beta \), while the other results do not follow the similar pattern. Fig. 11(a)–(d) show that the lateral buckling loads of web-tapered I-beams with concentrated load at the top flange of the free end section do not always decrease with tapering ratio, for instance, the lateral buckling loads of cantilevers with \( \beta \) equal to 0.1 are larger than those of prismatic beam (\( \beta \) equal to 1.0). This interesting phenomenon was generally considered to be caused by the combination of two opposite effects due to increase of \( \beta \) \cite{2}: (1) improvement of beam stiffness, which results in a higher buckling load; and (2) increase of the distance between the loading position and shear centre, which leads to a lower buckling load. However, the results shown in Fig. 13(a) and (b) indicate that this explanation is not completely true. In Fig. 13(a) and (b), the influences of loading position along the vertical axis of cross-section has been removed, since the concentrated load is always applied at the shear centre of the free end, while the lateral buckling loads of these beams, shown in Fig. 13(a) and (b), still do not increase with the tapering ratio (decrease of \( \beta \)) and the minimal buckling loads of cantilevers with a certain length is approximately at \( \beta \) equal to 0.45–0.6. This is due to the effects of the contributions of flanges resisting the vertical loads: part of the applied vertical load is resisted by the axial force of flanges along the longitudinal axes in their planes, besides the two opposite effects mentioned in \cite{2}.

(c) Lateral buckling of web-tapered simply supported beams subjected to concentrated load at mid-span

A simply supported beam with a concentrated loaded at the top flange of mid-span section, shown in Fig. 14, is considered. The parameters of this beam are \( E = 210 \) GPa, \( v = 0.3 \), \( h_L = 600 \) mm, \( t_f = 12.7 \) mm, \( t_w = 9.5 \) mm and \( b = 150 \) mm. The beam length \( L \) varies from 4 to 10 m, and the section ratio \( \beta \) from 0.1 to 1.0. The unreliable result \( Q^{AB}_{cr} \) is not included in the following comparisons.

Similar with comparisons for cantilevers, Fig. 15(a)–(d) also show that \( Q_{cr} \) are more accurate in assessment of the lateral buckling loads of web-tapered simply supported beams than \( Q^{Andrade}_{cr} \) compared to the reference results \( Q^{AS}_{cr} \), although obvious differences, due to effects of local deformations, exist between \( Q_{cr} \) and \( Q^{AS}_{cr} \) for beams of serious tapering.

For more comparisons, readers are referred to \cite{13}, in which lateral buckling of web-tapered I-beams of different loading conditions and load cases is investigated using different methods mentioned above. According to comparisons in \cite{13}, similar conclusions as in this paper can be drawn.

6. Conclusions

This paper reports a new theory on the lateral buckling of web-tapered I-beams. Deformations of web-tapered I-beams are carefully studied based on basic assumptions for thin-walled members, from which the relationships between displacements of the web and the flanges are created and the new equivalent sectional properties are presented. Subsequently, the total potential for lateral buckling analysis of web-tapered I-beams is proposed, with which the lateral buckling loads of these beams are obtained with the FE code developed. Lateral buckling loads of web-tapered cantilevers and simply supported beams of I sections based on the proposed total potential are compared with those from FE analyses using shell element models and beam elements of ANSYS as well as those from typical theories in existing literature. Comparisons and discussions show that the results based on the total potential of this paper are more accurate than existing theories in predicting the lateral buckling loads of web-tapered I-beams, indicating that the lateral buckling theory of web-tapered I-beam proposed in this paper is more rational than existing theories. Comparisons also show that the equivalent method using prismatic beam elements yield unreliable buckling loads of considered tapered beams. Needless to say, the development method of the theory presented in this paper can be extended to more general cases,
such as tapered beams of monosymmetric I-sections as well as other generic sections.

References
