Consider the BVP consisting of finding the function $u(x)$ satisfying

$$-\frac{d^2u}{dx^2} = 4200x^5$$

subject to the conditions $u(0) = u(1) = 0$.

a) Introduce a mesh consisting of two $P_1$ finite elements (i.e. linear local basis functions) and apply the Galerkin method to compute an approximate solution to the problem. Compare your results with those obtained from the exact solution.

b) Introduce a mesh consisting of one $P_2$ finite element (i.e. quadratic local basis functions) and apply the Galerkin method to compute an approximate solution to the problem. Compare your results with those obtained using two $P_1$ elements and with the exact solution.

c) Introduce a mesh consisting of $N$ $P_1$ finite elements (i.e. linear local basis functions) and apply the Galerkin method to compute an approximate solution to the problem. Compare your results with the exact solution.

2

A long, slender elastic beam (modulus $E$, constant cross sectional area $A$, length $L$, density $\rho$) is clamped to the ceiling at one end. The only load (per unit length) acting on the bar is the distributed force due to its own weight $\frac{mg}{L}$. Assuming static equilibrium, a force balance on a differential element of the bar yields the governing
equation for the longitudinal displacement $u(x)$ at each point as

$$-EA \frac{d^2u}{dx^2} = \frac{mg}{L}$$

or alternatively

$$-E \frac{d^2u}{dx^2} = \rho g$$

The beam is clamped to the ceiling at one end therefore $u(0) = 0$, and since the other end is free to move, $\frac{du}{dx}|_{x=L} = 0$.

a) Obtain the Ritz (minimum energy) variational formulation of the problem.

b) Obtain the Galerkin (weak) variational formulation of the problem.

c) Assume $E = 10^{11} \text{ N/m}^2$, $\rho = 8000 \text{ kg/m}^3$ (steel), $L = 10 \text{ m}$ and $A = 10^{-2} \text{ m}^2$. Introduce a mesh consisting of two $P_1$ finite elements (i.e. linear local basis functions) and apply the Galerkin method to compute approximations to the displacement, strain and stress along the beam. Compare your results with those obtained from the exact solution.

d) Introduce a mesh consisting of $N P_1$ finite elements (i.e. linear local basis functions) and apply the Galerkin method to compute approximations to the displacement, strain and stress along the beam. Compare your results with those obtained above and with the exact solution.

e) Now assume that a weight $W = 10^4 \text{ kg}$ is attached to the free end of the beam. The formulation of the problem is identical except that the boundary condition at that end where the weight is attached becomes

$$EA \frac{du}{dx}|_{x=L} = Wg$$

Carry out all the previously indicated computations for this case.

3

Consider the boundary value problem

$$-\frac{d^2u}{dx^2} + 10 \frac{du}{dx} - 10 = 0$$
subject to the conditions \( u(0) = u(1) = 0 \).

a) Introduce a mesh consisting of two \( P_1 \) finite elements (i.e. linear local basis functions) and apply the Galerkin method to compute an approximate solution to the problem. Compare your results with those obtained using one \( P_2 \) element and with the exact solution.

b) Introduce a mesh consisting of \( N \) \( P_1 \) finite elements (i.e. linear local basis functions) and apply the Galerkin method to compute an approximate solution to the problem. Compare your results with the exact solution.

4

Consider the boundary value problem consisting of determining the function \( u(x) \) satisfying

\[
-\frac{d}{dx}(k \frac{du}{dx}) = 0
\]

subject to the conditions \( u(0) = 0, u(1) = 1000 \)

The coefficient \( k = k(x) \) is a piecewise, step function of \( x \) such that

\[
k(x) = \begin{cases} 
50 & x \leq 0.75 \\ 
0.5 & x > 0.75 
\end{cases}
\]

a) Introduce a mesh consisting of four \( P_1 \) finite elements (i.e. linear local basis functions) and apply the Galerkin method to compute an approximate solution to the problem. Investigate the accuracy of your results by computing the values of various norms of the approximate solution as functions of the mesh size.

b) Represent the results obtained above in terms of element and global stiffness matrices \( K \) and forcing vector \( f \).

c) Introduce a mesh consisting of \( N \) \( P_1 \) finite elements (i.e. linear local basis functions) and apply the Galerkin method to compute an approximate solution to the problem. Compare your results with the exact solution.