Rapid low fidelity turbomachinery disk optimization

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ABSTRACT

Turbomachinery disks are heavy, highly stressed components used in gas turbines. Improved design of turbomachinery disks could yield a significant reduction in engine weight. This paper focuses on rapid low fidelity design and optimization of isotropic and transversely isotropic disks. Discussion includes the development of a one dimensional plane stress model, disk parameterization methods, and the implementation of a genetic algorithm for shape optimization. Three traditional geometry definition methods are compared to two new methods that are described and produce more optimum designs. Hardware from the GE E3 is used as an example. The analysis code is open-source, graphical, interactive, and portable on Windows, Linux, and Mac OS X.

Keywords:
Turbomachinery disk
Optimization
Plane stress
Flywheel

1. Introduction

High fuel costs and the tightening global economy have lead to a renewed push for reliable, highly fuel efficient gas turbine engines. Better engine flow path design and higher engine temperatures will inevitably lead to more efficient designs, but are not the only areas where research should be done. Methods for safely decreasing the overall engine weight must also be investigated. Turbomachinery disks comprise a large part of the structural weight of an engine, making them a perfect target for this investigation. Preferably, disk optimization should not greatly increase the amount of time needed for an engine design. This means that the general disk shape optimization should be completed early in the design process using quick, low fidelity models. Fine tuning of the shape will still be needed later in the design cycle, but proper optimization early on should ease this process. A fast disk optimization method may also be integrated into a larger component or system multi-disciplinary optimization approach.

This paper focuses on disk design and optimization using a one dimensional stress model. The derivation of the governing equations for isotropic and transversely isotropic disks will be described, followed by a description of the process used to discretize the model. Existing design codes often use parameterized disk geometry inputs to simplify the disk definition process [1,2]. Geometry and results using the common Ring, Web, and Hyperbolic parameterization methods will be compared. These three methods will also be compared to a new Continuous Slope (CS) parameterization and an arbitrary control point method. Optimization of the disk shape will be completed using a genetic algorithm with a specially tailored fitness function.

The limitations of the plane stress assumption will be discussed, along with methods to ensure the robustness of the resulting geometry. Further discussion will attempt to rate the effectiveness of each disk geometry definition method considering the weight of the resulting designs, the speed of optimization convergence, and the robustness of the final geometry. Future application of the stress model for the optimization of wound composite disks and flywheels will also be discussed. Throughout this paper hardware from the GE E3 turbofan engine will be used as an example [3,4]. Effective computer codes for use in engineering design must be robust, user friendly, and highly interactive. A disk analysis and optimization code has been created. The details of the program capabilities and the code design philosophy are described.

2. Analysis code

One dimensional stress models for disks of varying thickness have been in use for some time. Simple models for isotropic disks are available in many sources [5,6,1]. A small number of proprietary codes and commercial packages that implement these models exist. The disk design routine in GasTurb Details is the most widely available of these codes [2]. Free and publicly available disk optimization codes are not common. Code packages designed to support transversely isotropic (composite) materials and advanced geometry definition methods are even less common, even for proprietary applications. A complete disk analysis and optimization code package was needed and has been created. This code was intended to complement and to a small extent interface with the
T-Axi axisymmetric flow solver, available from the University of Cincinnati Gas Turbine Simulation Laboratory (GTSL) [7]. For continuity the name of the disk design code was simply chosen to be T-Axi Disk.

T-Axi Disk is a Graphical User Interface (GUI) based application written entirely in Fortran90 with calls to the DISLIN graphical libraries [8]. The code is open-source, freely available, and compatible with Windows, Linux, and Mac OS X operating systems. A version of this code has been released as an educational tool and is described by Gutzwiller et al. [9]. Fig. 1 shows a sample screenshot from the Windows release of T-Axi Disk. Downloads of the code distribution, example analyses, and complete documentation are on the GTSL website [7]. A summary of the code features are:

- Interactive design with an easy to use GUI.
- Detailed 2D and 3D stress contour plots.
- Rapid design optimization using a genetic algorithm.
- Automatic design tracking.
- Five disk definition methods.
- Support for isotropic and transversely isotropic materials.
- Temperature dependent material database with 10 common disk materials, compiled from trusted public sources [10,11].
- Automatic creation of axisymmetric Finite Element data for use with ANSYS (V11.0).
- Automatic blade and total dead weight estimation.

It may have been possible to complete the direct goals of this paper with a much simpler, command line based analysis code. Regardless, creating an easy to use, publicly available program is a worthwhile expenditure of time. One of the primary goals of any research is the transmission of knowledge and information throughout the engineering community. Typically this is done through journal publications, conference presentations, and other similar channels. Dispensing easy to use, well documented, and self contained software is another often overlooked way to transfer knowledge.

The main application of T-Axi Disk is as an optimization tool, and therefore needs to be able to execute thousands or millions of analyses without any code failures. Building a GUI around a code tends to expose any holes or bugs, leading to a very robust final product. Also, T-Axi Disk has the ability to output stress and geometry results at intermittent points in an optimization process. This feature allows the user to gain an understanding not only of what the optimized design looks like, but also how the algorithm finds the optimum. This type of understanding would be nearly impossible to acquire without a GUI based application. A specific design philosophy has been followed during the development of T-Axi Disk.

- **Open-source**

  T-Axi Disk is being released following the GNU General Public License [12]. This allows any end user to modify or expand the source code. Hopefully, this will prevent future engineers from having to recreate the basic analysis code, saving time which would be better spent on application and novel design studies.

- **Modularity**

  The T-Axi Disk source code was created with as much modularity as possible, allowing other researchers to easily extract subroutines and graphical procedures from the program.

- **Graphical User Interface**

  A clean GUI and well thought out presentation of results is very important. It significantly decreases the learning curve that comes with a new program and decreases the turnaround time for an analysis. The T-Axi Disk GUI was written using the DISLIN graphical libraries, which are very capable for this type of program [8].

- **Availability**

  T-Axi Disk is easy to obtain. The code distribution, source files, example analysis files, and documentation are all available for download from the University of Cincinnati GTSL website [7].

- **Ease of installation and portability**

  The entire T-Axi Disk distribution is contained within a single compressed file. Effort was taken to ensure that no unnecessary
dependencies were needed. Where possible, static linking has been used to ensure compatibility on a wide range of Windows, Linux, and Mac OS X systems. The Windows distribution of T-Axi Disk has a very simple installation process; the only step being the extraction of the files to a hard drive. Installation on Linux and Mac OS X requires the installation of DISLIN and the OpenMotif libraries, both of which are free and easy to acquire. All three operating systems use the same source files and Makefile, which should eliminate any platform specific confusion.

3. Governing equations for a transversely isotropic disk of arbitrary thickness

When developing an optimization problem, the solution speed of the simulation is very important. A one dimensional plane stress model was chosen due to its relative ease of setup and quick solution time. Previous work has shown that this type of model is very capable as a low fidelity stress predictor [1,9,6], although some caveats must be understood. The limitations of this model will be discussed in detail later in this paper. With any elasticity problem it is prudent to first take an inventory of any simplifying assumptions. For a rotating transversely isotropic disk of arbitrary thickness distribution the following can be assumed:

- Assume a right handed coordinate system.
  Axes: (r, θ, z).
  Displacements: (u, v, w).
- Assume plane stress in the r-θ plane.
  \[ \frac{d}{dr} = 0. \]
  \[ \tau_{r z} = \tau_{\theta z} = \sigma_z = 0. \]
- Assume axisymmetry.
  \[ \frac{d}{d\theta} = \nu = 0. \]

The plane stress equilibrium equations in terms of polar coordinates are:

\[
\begin{align*}
\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r \theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + B_r &= 0, \\
\frac{1}{r} \frac{\partial \sigma_\theta}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r \theta}}{\partial \theta} + 2 \frac{\tau_{r \theta}}{r} + B_\theta &= 0.
\end{align*}
\]

(1)

Application of the axisymmetric assumption and the inertial body forces leads to a simplified relation:

\[
\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \rho \omega^2 r = 0.
\]

(2)

The inclusion of an arbitrarily varying thickness distribution leads to the following relation, as shown and derived in some advanced elasticity books [5].

\[
\frac{d}{dr} (\text{tr} \sigma_r) - \tau_{r \theta} + \rho \omega^2 r^2 = 0.
\]

(3)

Application of the axisymmetric assumption to the standard strain-displacement relations leads to the simplified expressions below [13]. As expected, the shear strain drops to zero.

\[
\begin{align*}
\epsilon_r &= \frac{du}{dr} \\
\epsilon_\theta &= \frac{u}{r} \\
\tau_{r \theta} &= 0
\end{align*}
\]

(4)

The stress strain relations for this problem were derived from the most general form of Hooke’s law. For this problem the total stress has both a mechanical and a thermal component.

\[
\begin{bmatrix}
\sigma_r \\
\sigma_\theta \\
\sigma_z \\
\tau_{r z} \\
\tau_{r \theta} \\
\tau_{r \theta}
\end{bmatrix} =
\begin{bmatrix}
\sigma_r^m \\
\sigma_\theta^m \\
\sigma_z^m \\
\tau_{r z}^m \\
\tau_{r \theta}^m \\
\tau_{r \theta}^m
\end{bmatrix} +
\begin{bmatrix}
\sigma_r^t \\
\sigma_\theta^t \\
\sigma_z^t \\
\tau_{r z}^t \\
\tau_{r \theta}^t \\
\tau_{r \theta}^t
\end{bmatrix}.
\]

(5)

The stiffness tensor relating the stresses to the strains in its most general form is shown below.

\[
[C] =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}.
\]

(6)

The mechanical and thermal stresses are now defined by Hooke’s law and the strain-displacement relations [14].

\[
\begin{bmatrix}
\sigma_r^m \\
\sigma_\theta^m \\
\sigma_z^m \\
\tau_{r z}^m \\
\tau_{r \theta}^m \\
\tau_{r \theta}^m
\end{bmatrix} =
[C] \begin{bmatrix}
\epsilon_r \\
\epsilon_\theta \\
\epsilon_z \\
\gamma_{r z} \\
\gamma_{r \theta} \\
\gamma_{r \theta}
\end{bmatrix}.
\]

(7)

\[
\begin{bmatrix}
\sigma_r^t \\
\sigma_\theta^t \\
\sigma_z^t \\
\tau_{r z}^t \\
\tau_{r \theta}^t \\
\tau_{r \theta}^t
\end{bmatrix} =
[B] \begin{bmatrix}
\beta_r \\
\beta_\theta \\
\beta_z \\
\beta_{r z} \\
\beta_{r \theta} \\
\beta_{r \theta}
\end{bmatrix}.
\]

(8)

(9)

(10)

The \( \epsilon_r \) and \( \beta_r \) terms come from the assumption of plane stress. It is assumed that there are no stresses in the z, or out-of-plane, direction. For this to be possible the strain in the z direction must be non-zero. Setting \( \sigma_z = \sigma_r^t = 0 \) yields the expressions for \( \epsilon_r \) and \( \beta_r \). Combining Eqs. 5, 7, 8 yields the stresses in a rotating disk as a function of the radial displacement:

\[
\begin{align*}
\sigma_r &= A \frac{du}{dr} + B \frac{u}{r} - \mathcal{A} \mathcal{B} T - \mathcal{B} \mathcal{A} T, \\
\sigma_\theta &= B \frac{du}{dr} + D \frac{u}{r} - \mathcal{B} \mathcal{B} T - \mathcal{D} \mathcal{A} T,
\end{align*}
\]

with the substitutions for \( A, B, D \)

\[
\begin{align*}
A &= \left( C_{11} C_{33} - C_{13}^2 \right) \\
B &= \left( C_{12} C_{33} - C_{13} C_{23} \right) \\
D &= \left( C_{22} C_{33} - C_{23}^2 \right)
\end{align*}
\]

(11)

(12)

(13)

and the stiffness constants for a transversely isotropic material
\[ C_{11} = C_{33} = \left( \frac{E_{r}(E_{r} - E_{\theta})}{E_{r} - 2E_{r}v_{r\theta}} \right) \]

\[ C_{22} = \left( \frac{E_{\theta}}{E_{r} - 2E_{r}v_{r\theta}} \right) \]

\[ C_{12} = C_{23} = \left( \frac{E_{r}E_{\theta}}{E_{r} - 2E_{r}v_{r\theta}} \right) \]

\[ C_{13} = \left( \frac{E_{r}^{2}v_{r\theta}}{E_{r} - 2E_{r}v_{r\theta}} \right) \]

Simplifying the expressions for isotropic materials \((E_{r} - E_{\theta},\; v_{r\theta} = 0)\) leads to the familiar disk stress equations shown in many sources [1,5].

4. Solution process

As derived in the previous section, the constitutive equations for a rotating anisotropic disk can be written as shown in Eqs. (11) and (12). The axisymmetric plane stress equilibrium equation is also shown in Eq. (3). Assuming the general case of a disk with a hole, loaded either at the inner radius (bore) or the outer radius (rim), the following boundary conditions can be assumed. In both loading cases the effect of the dead weight is applied as either a compressive or tensile radial stress.

Boundary condition 1: at the bore

\[ \sigma_{r} = \frac{A}{d} \frac{du}{dr} + \frac{B}{r} - A\frac{d\theta}{dt} - B\frac{\theta}{r} = -\frac{n_{0}m_{0}r\Omega^{2}}{2\pi}\sigma_{r} \cdot \Delta r. \]  \hspace{1cm} (15)

Boundary condition 2: at the rim

\[ \sigma_{r} = \frac{A}{d} \frac{du}{dr} + \frac{B}{r} - A\frac{d\theta}{dt} - B\frac{\theta}{r} = -n_{0}m_{0}r\Omega^{2} \cdot \Delta r. \]  \hspace{1cm} (16)

In a typical bladed disk application the disk is loaded on the outer rim, which means that boundary condition #2 is applied with a positive sign tensile stress. In this case there is no loading on the bore, so \(m_{0}\) and \(\sigma_{r}\) in boundary condition #1 are effectively zero. The two boundary conditions are presented in a more general form so novel applications of bore loaded external disks can be explored.

This boundary value problem can be solved using a variety of methods. The method chosen for this application involves developing an expression for the radial stress \((\sigma_{r})\) dependent on the radial displacement at a series of three contiguous points. Integrals are approximated using an averaging scheme similar to the box method presented by Keller and Cebeci [15]. This method has a number of distinct advantages over other discretization methods, shown in detail in the two appendices at the end of this paper. In summary, the disk is discretized along \(n\) radial stations leading to a system of \((n - 2)\) distinct equations, similar to what a central difference formula would produce. Two more equations are needed to complete the system and can be found from the known radial stress boundary conditions at the disk bore and rim. A forward or backwards difference method is used at the boundaries to generate the remaining two equations. The end result of this process is a \((n) \times (n)\) matrix equation with the radial displacement as the unknown variable. The matrix is tridiagonal except for the first and last equations; a Thomas algorithm [16] is modified accordingly for a quick solution. Once the radial displacement at each point is found the radial and tangential stresses at each point may be calculated using Eqs. (11) and (12). Once the stresses are known at each radial station, empirical design criteria may be calculated. One common design criteria, the burst margin [1], is defined as:

\[ BM = \frac{0.47\sigma_{ult}}{\sigma_{marg}} \]  \hspace{1cm} (17)

5. Stress model validation

The stress model implementation in T-Axi Disk was validated by comparison to closed form solutions for both the isotropic and anisotropic cases [17]. This simplified test case is a constant thickness ring spun at 5000 RPM with other details shown in Table 1. It was assumed that no dead weight was applied in this case, which leads to a zero state of radial stress at both the bore and the rim. The results of this comparison for a disk constructed out of Inconel718 and Epoxy–Fiberglass are shown in Figs. 2 and 3, respectively. For the anisotropic case the material has a degree of anisotropy of approximately 4.5. The results of this test show a close correlation between the closed form solution and the discretized solution. The T-Axi Disk solutions shown in Figs. 2 and 3 use 100 discretization points. With a larger number of points the closed form and discretized method converge to the same solution, but the run time is proportionately longer. The result of a convergence study for the isotropic test case with 11–1001 points is shown in Fig. 4. The error in the midpoint radial displacement and the grid spacing \((\Delta r)\) are plotted on a Log–Log scale. The slope of the resulting line is 1.93, which indicates that the method is second order accurate. The 100 point solution reports displacements and stresses within 0.1% of the closed form solution, which for a low fidelity analysis has been assessed as sufficient both in accuracy and solution time.

This validation case is simple, but it illustrates the important differences between a common isotropic and anisotropic material. This case has no dead weight applied, meaning the stresses are due to the mass of the spinning disk. The isotropic Inconel718 has a density nearly four times greater than the anisotropic Epoxy–Fiberglass composite material. Careful investigation of Figs. 2 and 3 shows that the maximum tangential stress is much higher in the Inconel718 case, which is consistent with the expected trends.

6. Disk definition methods

When optimizing a component, it is necessary to have the geometry defined using a finite number of modifiable design parameters. In this investigation five geometry parameterization

| Table 1 |
| Code validation test case. |
| **Isotropic material** | Inconel718 |
| Anisotropic material | Epoxy–Fiberglass |
| Angular speed | 5000 RPM |
| Bore radius | 0.4 m |
| Rim radius | 1.0 m |
| Disk temperature | 20.0 °C |
methods were used. Representative disks defined by four of the methods are shown in Fig. 5. The main purpose of these parameterization methods is to define the live weight shown in Fig. 5. Permutations of the Ring, Web, and Hyperbolic disk definition methods have been previously used in other disk design programs [1,2]. T-Axi Disk also uses these methods along with the Continuous Slope (CS) parameterization, which was developed specifically for this investigation. A much more detailed description of these geometry definitions is included in the T-Axi Disk documentation.

6.1. Common parameterization methods

The Ring parameterization is by far the simplest disk definition method. Only one parameter; the bore radius, is needed to define the live weight of this type of disk. The constant disk thickness and rim radius are defined by the blade and blade root geometry, which is held static throughout any optimization process. This type of disk is inappropriate for most highly loaded turbomachinery applications, but may be feasible for use with composite disks in fans and early booster stages.

In many turbomachinery applications the Web parameterization could be safely viewed as a “standard” definition method. Many existing compressor and turbine disk designs can be roughly recreated using the Web parameterization. Five parameters defining the bore radius, bore width, inner rim height, web thickness, and outer rim height are needed. Once again the rim radius and width are determined from the blade geometry. The inner rim is connected to the web by a geometry segment at a slope of 45° while the outer rim is connected to the web with a segment at a slope of 60°. This seemingly arbitrary connection method both limits the number of input parameters and helps ensure that the final geometry upholds the plane stress assumption.

The Hyperbolic parameterization is similar to the Web method described above. Once again the bore radius, bore width, inner rim height, web thickness, and outer rim height must all be specified. However, the connection between the inner rim and the web is no longer a simple linear segment. An additional parameter, the disk shape factor ($dsf$), is needed to define the thickness profile.

At any radial station $i$ in this region

$$t_i = t_2 + \left( t_1 - t_2 \right) \frac{r_i - r_1}{r_2 - r_1} \cdot dsf,$$  \hspace{1cm} (18)

where station 1 is at the edge of the inner rim and station 2 is at the thinnest point of the web. When paired with a plane stress model this disk definition method may produce passing disk designs with very low masses. However, extra care should be taken to ensure that the geometry does not violate the plane stress assumption.

6.2. Continuous Slope (CS) parameterization

Like the Hyperbolic method, the live weight in disks defined by the CS parameterization is controlled by six input parameters. The definition of the dead weight is the same as with the Web and Hyperbolic methods, but the similarities end there. The CS disk geometry is broken up radially into three segments: the inner
rim, the web, and the outer rim. The inner and outer rim are modeled with fourth order line segments while the web region is modeled with a linear line segment. With the exception of the disk rim width, all of the inputs for the live weight are non-dimensionalized. Non-dimensionalization of the geometry parameters creates a more robust design space ideal for use in an optimization routine.

6.2.1. Static inputs

The static inputs (A–E) define the dead weight geometry. Please reference the T-Axi Disk documentation for a detailed description of these inputs.

6.2.2. Live weight design parameters

- (F) Disk bore radius

The disk bore radius is set as a ratio of the disk rim radius. This parameter is defined by the T-Axi Disk input PR1, with possible values between 0.0 and 1.0.

- (G) Inner web radius

The radius of the inner web is set as a ratio of the disk rim radius minus the bore radius. This parameter is defined by the T-Axi Disk input PR2, with possible values once again being between 0.0 and 1.0. This input method ensures that the radius of the disk bore will never exceed the inner web radius. Also, setting the lower bound for this value to a conservative number such as 0.2 is a very easy way to ensure that the resulting design does not break the plane stress assumption.

\[ r = C_1 r^4 + C_2 r^3 + C_3 r^2 + C_4 r + C_5 \]  

(19)

The five coefficients \(C_1–C_5\) define the general shape of the curve and can be found from the prescribed slope of the line at the two end points and the known value of \(t\) at each control point. As shown in Fig. 6, two of the control points at the disk bore and the inner web have already been defined. The third control point is assumed to be located at the radial midpoint between these two points. The thickness of the disk at the middle control point is specified in a non-dimensionalized form as T-Axi Disk input S12. Typically this input can have values between 0.0 and 1.0, with 0.0 being the case where the thickness at the middle control point equals the thickness at the inner web and 1.0 being the case where the thickness at the middle control point equals the thickness at the disk bore. Experience has found that values between 0.2 and 0.8 produce the most robust geometry. The slope at the inner web location is set to match the slope of the linear web section. The slope at the bore is set to the slope of the linear line through the bore and middle control points. Effectively, this means that the slope at the bore is also controlled by the input parameter S12.

- (M,N) Outer rim definition

To decrease the amount of input parameters the definition of the outer rim is set using a number of parameters that are hard-coded, and not user defined. The thickness profile is defined using a fourth order line segment fit through three control points, with the first and last control points located at the disk rim and at the outer web radius. The radius of the outer web control point is automatically set 95% of the disk rim radius. The third control point is assumed to be located exactly midway between the two other points, both in radius and thickness. The slope of this line at the outer radius is assumed to be zero. The slope of the line at the outer web control point is matched to the slope of the linear web segment. This definition is essentially a simplified version of the method used to model the inner rim region.

The rate of change of the thickness profile is matched at all control points, leading to a final disk profile that is smooth and continuous, unlike the web and hyperbolic methods that produce sharp corners. If a higher fidelity follow up analysis is performed these sharp corners may lead to problems such as stress singularities.

6.3. Arbitrary control point method

In addition to the four parameterizations mentioned above, T-Axi Disk also allows for creation of disks using an arbitrary number of control points, evenly spaced between the bore and the rim. At each control point the disk thickness may be specified. The thickness distribution between each control point is linear. This method was developed primarily for use with an automated optimization routine. Depending on the number of control points this method could have a very large number of parameters, making design optimization much more time intensive. However, this geometry definition method is the most general and should be very useful as a comparison against the other parameterization methods. This
method may also be used to refine designs created using a simpler parameterization method.

7. Optimization procedure

A genetic algorithm was written for the optimization of the disk profile. The algorithm used in this paper was not based on any specific source, but follows the same basic optimization procedure shown in a wide range of sources [18, 19]. Genetic algorithms are very good at finding global optimum design points, which makes them very useful when dealing with complex optimization profiles with many potential local minima. However, these algorithms may require a large number of iterations for complete convergence. The solution of the disk stress equations is quick enough that the extra time needed for convergence is acceptable. A similar genetic algorithm approach has been used by Eby et al. in the optimization of simple flywheel shapes [20]. A gradient based optimization algorithm was considered for this research, but local minima experienced when using some of the parameterizations became a problem. Also, gradient based optimization is inappropriate for complex composite or hybrid designs.

7.1. Fitness function

All optimization methods require a clearly defined fitness function. The main goal of the disk profile optimization is to minimize weight, but it must ensure that the disk meets a target stress factor of safety and meets any geometry limitations and requirements. The optimization algorithm in T-Axi Disk attempts to maximize the fitness function, so the primary term in the fitness function is the inverse of the disk mass.

\[
F = \frac{1}{m} - \frac{L_{str}}{m} \frac{F_{str}}{m} - \frac{L_{curv}}{m} \frac{F_{curv}}{m}
\]

(20)

The \( F_{str} \) term accounts for the stress requirements of the disk. In a passing disk design this term will be zero, but will increase as sections of the disk experience stresses above the target level. \( L_{str} \) is a Lagrange multiplier that allows more control over the shape of the optimization space. Increasing \( L_{str} \) increases the fitness penalty for breaking the stress requirements, which may lead to instability in the optimization process. Using excessively low values for \( L_{str} \) may result in optimized designs that fail to meet the stress factor of safety.

Similarly, the \( F_{curv} \) and \( L_{curv} \) terms apply a penalty to any design that breaks a specified curvature requirement. This term is only used for the arbitrary control point definition method. It effectively guides the optimization to a smooth final design without any rapid changes in section thickness. Without this control the optimization would lead to an erroneous design that completely violates the plane stress assumption, as shown in the discussion section of this paper.

7.2. Algorithm features

The genetic algorithm included with T-Axi Disk uses a fairly simple recombination and random mutation procedure for a specified population size over a specified number of generations. The maximum allowable mutation as a percentage of the parameter space may be specified. Also included are a number of modifiable controls to aid convergence for a wide range of problems. A simple elitism capability allows the user to specify a percentage of the best designs in a population that are omitted from the recombination step, effectively forcing a local search.

A variable mutation rate has also been included as an option for the arbitrary control point optimization. This feature relates the maximum allowable mutation to the rate of change of the fitness function. As the fitness function nears its optimized value the mutation rate decreases, which allows the algorithm to easily fine tune the design.

Convergence control for the arbitrary control point method is simple. If no increases in fitness are found over a specified number of populations, the function is assumed to be converged. For the four other parameterization methods the solution reaches a stable state in only a few seconds, so for simplicity the convergence is ensured by running the algorithm for a known number of generations.

8. Example analyses: \( E^3 \) compressor third stage

8.1. Problem setup

The GE Energy Efficient Engine (\( E^3 \)) high pressure compressor (HPC), shown in Fig. 7, was designed in the late 1970s. Eventually many features of this design were used in the GE90 turbofan. Many of the documents and designs from this project are publicly available, making the \( E^3 \) a perfect test case for verifying results from a new program or for performing a comparative study. The third stage disk has been chosen as the test case for this investigation.

Table 2 shows the inputs that were used to build the T-Axi Disk input file. The majority of these values were taken or derived from the available \( E^3 \) documents [4, 3]. The minimum bore radius and maximum section width parameters shown in Table 2 are assumed values that have been determined from the \( E^3 \) cross section drawing. Turbomachinery components must be designed in a way that accounts for the physical assembly of the entire engine. To meet this requirement, parameter bounds of 0.09 m for the minimum bore radius and 0.05 m for the maximum section thickness were assumed.

8.2. Optimization results

The optimization problem described above was solved using T-Axi Disk and the five geometry definition methods. It should be noted that an isotropic disk with a Ring parameterization is entirely inappropriate for this application. Neither manual inspection or the genetic algorithm yielded any passing ring designs, so the

[Image](315x193 to 559x268)

**Fig. 7.** \( E^3 \) 10 stage compressor cross section.

<table>
<thead>
<tr>
<th>Table 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disk material</td>
</tr>
<tr>
<td>Disk rim radius</td>
</tr>
<tr>
<td>Disk rim thickness</td>
</tr>
<tr>
<td>Maximum allowable section width</td>
</tr>
<tr>
<td>Minimum allowable bore radius</td>
</tr>
<tr>
<td>Dead weight CG</td>
</tr>
<tr>
<td>Total dead weight</td>
</tr>
<tr>
<td>Disk temperature</td>
</tr>
<tr>
<td>Temperature adder</td>
</tr>
<tr>
<td>Angular speed</td>
</tr>
</tbody>
</table>
ring parameterization will not be included in this comparison. The optimized profiles and the corresponding disk masses for the other four methods are shown in Fig. 8 and Table 3. Stress plots for each profile are also shown in Fig. 8. The solid line represents the Von Mises stress at each radial station while the dashed line represents the stress factor of safety line. A constant population size of 100 individuals was used as an input for the genetic algorithm. The Web, Hyperbolic, and CS design optimizations were run for a total of 300 populations. Each optimization was run multiple times with a random starting population to ensure convergence. The arbitrary control point design was assumed to have 20 control points. The solution was assumed to be converged when 3000 populations had passed without an increase in the design fitness.

9. Discussion

9.1. Comments on disk parameterization

Careful investigation of the results yields some very useful information. The nature of mass optimization boils down to minimizing the excess stress margins at every location in the body, decreasing the weight in the process. When applied to a 1D disk stress model this is effectively the same as finding a design in which the Von Mises stress line has a shape and magnitude similar to the safety factor line. Disk parameterization and optimization methods should be designed with this simple concept in mind.

As expected, the arbitrary control point method produced the lightest disk. The CS method was the next lightest, followed by the Web and Hyperbolic methods. In the Web parameterization it is assumed that the web region of the disk is of constant thickness. This leads to a significant amount of extra material and extra stress margin at the outer radial stations. The addition of one more parameter allowing linear variation of the web thickness would greatly increase the usefulness of the Web parameterization method.

The Hyperbolic parameterization has a variable web thickness, but other problems occur when using this method. For this analysis a lower bound of 0.2 was set for the disk shape factor. Fully opening up this parameter's design space yields a lighter weight disk than the 7.07 kg design shown in Fig. 8. This design, with a disk shape factor of 0.02 and a mass of 5.54 kg, is shown in Fig. 9. However, investigation with higher fidelity finite element models has shown that this design is a false minimum; in reality the unconstrained Hyperbolic design experiences a significant amount of out-of-plane stresses, leading to much higher Von Mises Stresses. The results of an investigation of the rejected and accepted Hyperbolic geometries with ANSYS V11.0 are shown in Table 4. Assuming a lower bound of 0.2 for the disk shape factor led to a result with reasonably low out-of-plane stresses, with maximum values near 5% of the corresponding in-plane stresses. The rejected Hyperbolic disk design experiences out-of-plane stresses with magnitudes over 44% of the in-plane stresses.

What constitutes adequate fulfillment of the plane stress assumption is a very subjective topic that must be decided on a case by case basis. Regardless, even for a low fidelity analysis it is unacceptable to use a design that neglects a stress that is nearly half of the assumed maximum. When optimizing turbomachinery disks with a plane stress model, "flange" type designs like the one shown in Fig. 9 grossly under predict the overall state of stress and must be avoided. Any useful disk optimization routine must have intelligent bounds on the parameter design space or a method of curvature control to prevent this type of design. In addition to the options shown in this paper, a number of alternative methods to account for the out-of-plane stress have been presented by Genta and Bassani [21]. Some previously published papers and

---

Table 3

<table>
<thead>
<tr>
<th>Method</th>
<th>Run time (s)</th>
<th>Disk mass (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Web</td>
<td>9.02</td>
<td>6.07</td>
</tr>
<tr>
<td>Hyperbolic</td>
<td>9.43</td>
<td>7.07</td>
</tr>
<tr>
<td>CS</td>
<td>9.46</td>
<td>5.46</td>
</tr>
<tr>
<td>Arbitrary control point</td>
<td>456.10</td>
<td>5.19</td>
</tr>
</tbody>
</table>

---

Fig. 8. $E^3$ third stage optimized disk profiles.

Fig. 9. Rejected hyperbolic disk design.
textbooks on turbomachinery design optimization have ignored this facet of the problem, and should be referenced with caution [6, 22].

Of the simple parameterized geometry methods, the CS method was the most successful. Intelligent bounds on the input parameters once again were used to prevent any large out-of-plane stresses. Comparison of the continuous slope design to the arbitrary control point design shows a close similarity in overall shape. From these results it may be assumed that for an isotropic disk with a small temperature gradient, a linear definition of the web region is sufficient. The fourth order spline definition of the inner rim region also proved to be satisfactory.

The Web, Hyperbolic, and CS designs were all optimized over 300 populations, so the closeness in total run time for the three is to be expected. The arbitrary control point method took much longer to converge; totaling 19,679 populations for a run time of 456.1 s. Even this amount of time is not unreasonable, especially considering that a full design space study using 20 control points was the most successful. Intelligent bounds on the input parameters and the same level of curvature control with 10, 15, 20, and 25 control points. The results in Table 5 and Fig. 10 clearly show that using more than 20 control points is unnecessary. The optimized 10 control point design is very rough, with the inner rim region modeled in a nearly linear manner and the bore radius optimizing to a larger value than expected. The 15 control point design is better, but there is still not enough resolution in the inner rim region. The optimized 20 and 25 control point designs are very similar, and for a low fidelity initial analysis any extra level of detail is unnecessary.

The results of $E^3$ investigation indicate that for a simple disk optimization problem with isotropic materials and a small or nonexistent temperature gradient the arbitrary control point method is of little use. The small decreases in disk weight from this approach do not offset the extra time needed for a converged solution. Also, the higher fidelity axisymmetric or full 3D analysis that inevitably follows any initial design work will probably lead to design changes that negate any small weight gains. However, it is difficult to predict how the Web, Hyperbolic, and CS parameterization methods will perform when tied to a more complex system. Transversely isotropic designs, such as one built from a wound composite material, or a disk with a large temperature gradient may be especially problematic. The extra robustness and wider design space of the arbitrary control point method could be very useful in these more complex cases.

### Table 4

<table>
<thead>
<tr>
<th>Hyperbolic disk – out-of-plane stress comparison.</th>
<th>Rejected disk</th>
<th>Accepted disk</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{\text{max}}$ in-plane (MPa)</td>
<td>421</td>
<td>425</td>
</tr>
<tr>
<td>$\sigma_{\text{max}}$ out-of-plane (MPa)</td>
<td>186</td>
<td>25</td>
</tr>
<tr>
<td>Out/in plane (%)</td>
<td>44.2</td>
<td>5.7</td>
</tr>
</tbody>
</table>

### Table 5

<table>
<thead>
<tr>
<th>Number of points</th>
<th>Disk mass (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5.56</td>
</tr>
<tr>
<td>15</td>
<td>5.22</td>
</tr>
<tr>
<td>20</td>
<td>5.19</td>
</tr>
<tr>
<td>25</td>
<td>5.19</td>
</tr>
</tbody>
</table>

9.2. Comments on the arbitrary control point method

The $E^3$ problem shown in the previous section used 20 points with the arbitrary control point method. A study was performed to ensure that this number of points was sufficient. The optimization process was completed four times using the same genetic algorithm inputs and the same level of curvature control with 10, 15, 20, and 25 control points. The results in Table 5 and Fig. 10 clearly show that using more than 20 control points is unnecessary. The optimized 10 control point design is very rough, with the inner rim region modeled in a nearly linear manner and the bore radius optimizing to a larger value than expected. The 15 control point design is better, but there is still not enough resolution in the inner rim region. The optimized 20 and 25 control point designs are very similar, and for a low fidelity initial analysis any extra level of detail is unnecessary.

The results of $E^3$ indicate that for a simple disk optimization problem with isotropic materials and a small or nonexistent temperature gradient the arbitrary control point method is of little use. The small decreases in disk weight from this approach do not offset the extra time needed for a converged solution. Also, the higher fidelity axisymmetric or full 3D analysis that inevitably follows any initial design work will probably lead to design changes that negate any small weight gains. However, it is difficult to predict how the Web, Hyperbolic, and CS parameterization methods will perform when tied to a more complex system. Transversely isotropic designs, such as one built from a wound composite material, or a disk with a large temperature gradient may be especially problematic. The extra robustness and wider design space of the arbitrary control point method could be very useful in these more complex cases.

### 10. Conclusions

The development and implementation of a low fidelity stress model and rapid optimization procedure for turbomachinery disks has been presented. T-Axi Disk, an open source disk analysis and optimization code, has been created and is freely available [7, 9]. The code is graphical, interactive, robust, and portable on Windows, Linux, and Mac OS X. The Continuous Slope disk parameterization method has been introduced as a viable or in some cases superior alternative to the existing Web, Ring, and Hyperbolic methods. The arbitrary control point method has also been introduced as a more robust and general disk definition method which may prove useful for the optimization of complex disk systems.

A plane stress model was used in this investigation due to its speed and overall accuracy. Problems with the model stemming from the omission of the out-of-plane stress were discussed and averted. It is recommended that any disk design and optimization procedures implement some form of intelligent parameter bounds or a method of curvature control to avoid any erroneous designs.

The examples in this paper were restricted to a relatively simple case using isotropic materials without an applied thermal gradient. The model was designed to be very robust, so future investigations including advanced materials and alternative loading methods are possible. The arbitrary control point geometry definition method may prove to be very useful when optimizing disks with large thermal loads or for optimization of wound composite designs. One application of the model would be for the shape optimization of next generation composite flywheels. The CS parameterization method presented in this paper would also be very useful in this case. The addition of three more parameters allowing for complete...
modification of the outer rim would create a very robust design space that may contain a good approximation of the optimum flywheel design.

Lastly, the calculations presented in this paper are all very fast. The quick solution time means that integration into a larger, multi-disciplinary gas turbine optimization project is possible. Upcoming work will focus on coupling the disk optimization method with flow path optimization using a quick axisymmetric solver, such as T-Axi [7,23]. Optimization will focus on the maximization of a fitness function that includes both weight and efficiency terms. The final output of future work would be a self contained design package that allows for the rapid low fidelity optimization of an entire compressor or turbine system, allowing designers to quickly obtain a rough design for feasibility studies or as a bridge to more detailed, higher fidelity analysis.

Acknowledgment

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Appendix A. Disk discretization numerics

A.1. Governing equations and substitutions

The following derivation and discretization is valid for the most general case with transversely isotropic materials and an arbitrary radial temperature profile. The axisymmetric plane stress equilibrium equation is

\[
\frac{d}{dr}(tr \sigma_r) - t \sigma_o + t \rho \omega^2 r^2 = 0. \tag{21}
\]

The constitutive equations for a rotating anisotropic disk are

\[
\sigma_r = \frac{A}{r} \frac{du}{dr} + B \frac{u}{r} - A \alpha_T - B \alpha_r T, \tag{22}
\]

\[
\sigma_o = B \frac{u}{r} + D \frac{1}{r} - B \alpha_r T - D \alpha_o T, \tag{23}
\]

where

\[
A = \left( \frac{C_{11} C_{33} - C_{12}^2}{C_{33}} \right)
\]

\[
B = \left( \frac{C_{12} C_{33} - C_{11} C_{22}}{C_{33}} \right)
\]

\[
D = \left( \frac{C_{22} C_{33} - C_{23}^2}{C_{33}} \right)
\]

\[
C_{11} = C_{33} = \left( \frac{E_1 (E_0 - 2E_3 v_p)}{E_0 - 2E_3 v_p^2} \right)
\]

\[
C_{12} = C_{23} = \left( \frac{E_1 E_3 v_p}{E_0 - 2E_3 v_p^2} \right)
\]

\[
C_{13} = \left( \frac{E_2 E_3 v_p}{E_0 - 2E_3 v_p^2} \right)
\]

Assuming the general case of a disk with a hole, loaded either at the bore or rim, the boundary conditions are

Boundary condition 1: at the bore

\[
\sigma_r = \frac{A}{r} \frac{du}{dr} + B \frac{u}{r} - A \alpha_T - B \alpha_r T = -\frac{n_b m_b r^2}{2 \pi r t} \omega^2. \tag{26}
\]

Boundary condition 2: at the rim

\[
\sigma_r = \frac{A}{dr} + B \frac{u}{r} - A \alpha_T - B \alpha_r T = \frac{n_b m_b r^2}{2 \pi r t} \omega^2. \tag{27}
\]

In both loading cases the effect of the dead weight is applied as either a compressive or tensile radial stress. In a typical bladed disk application the disk is loaded on the outer rim, which means that boundary condition #2 is applied with a positive sign tensile stress. In this case there is no loading on the bore, so \( m_b \) and \( \sigma_r \) in boundary condition #1 are effectively zero. The two boundary conditions are presented in a more general form so novel applications of bore loaded external disks can be explored.

A.2. Discretization of the equations in the continuum

All of the calculations in this section use the following substitutions, shown with dummy variable \( X \).

\[
\Delta X_{21} = X_{i+1} - X_i \quad \Delta X_{21} = \frac{X_{i+1} + X_i}{2} \tag{28}
\]

\[
\Delta X_{10} = X_i - X_{i-1} \quad \Delta X_{10} = \frac{X_i + X_{i-1}}{2} \tag{29}
\]

This boundary value problem can be solved using a variety of methods. The method chosen involves developing an expression for the radial stress \( \sigma_r \) dependent on the radial displacement at a series of three contiguous points. This process leads to \((n - 2)\) distinct equations, similar to what a central difference formula would produce. Two more equations complete the system and can be found from the known radial stress boundary conditions at the disk bore and rim. A forward or backward difference method is used at the boundaries to generate the remaining two equations.

Starting with Eq. (22), the first step is to estimate \( \sigma_r \), over intervals [1–2] and [0–1].

\[
\sigma_{i+1} = \frac{\Delta \sigma_{i+1}}{2} = \frac{\Delta X_{21}}{\Delta X_{10}} \frac{\Delta \sigma_{i+1}}{\Delta X_{21}} = \frac{\Delta X_{21}}{\Delta X_{10}} \frac{\Delta \sigma_{i+1}}{\Delta X_{21}} \frac{B_{21}}{\Delta X_{21}} - \frac{B_{21}}{\Delta X_{21}} \frac{\Delta \sigma_{i+1}}{\Delta X_{21}} \frac{B_{21}}{\Delta X_{21}}. \tag{30}
\]

\[
\sigma_{i+1} = \frac{\Delta \sigma_{i+1} + \sigma_{i+1}}{2} = \frac{\Delta X_{21}}{\Delta X_{10}} \frac{\Delta \sigma_{i+1}}{\Delta X_{21}} = \frac{\Delta X_{21}}{\Delta X_{10}} \frac{\Delta \sigma_{i+1}}{\Delta X_{21}} \frac{B_{21}}{\Delta X_{21}} - \frac{B_{21}}{\Delta X_{21}} \frac{\Delta \sigma_{i+1}}{\Delta X_{21}} \frac{B_{21}}{\Delta X_{21}}. \tag{31}
\]

Next, integrate Eqs. (21) and (23) over intervals [1–2] and [0–1].

\[
\int_1^2 \frac{d}{dr} (tr \sigma_r) \, dr = \int_1^2 (t \sigma_r - t \rho \omega^2 r^2) \, dr
\]

\[
(t \sigma_r)_2 - (t \sigma_r)_1 = t \sigma_r \Delta r_{21} - t \sigma_r \Delta r_{10} = t \sigma_r \Delta r_{21} \frac{r_2^2 - r_1^2}{3}. \tag{32}
\]

\[
\int_0^1 \frac{d}{dr} (tr \sigma_r) \, dr = \int_0^1 (t \sigma_r - t \rho \omega^2 r^2) \, dr
\]

\[
(t \sigma_r)_1 - (t \sigma_r)_0 = t \sigma_r \Delta r_{10} - t \sigma_r \Delta r_{10} = t \sigma_r \Delta r_{10} \frac{r_1^2 - r_0^2}{3}. \tag{33}
\]

Fig. 11. Disk discretization.
\[
\int_0^1 \sigma \, dr = \int_0^1 B \frac{du}{dr} \, dr + \int_0^1 D_C \frac{u}{r} \, dr - \int_0^1 B \frac{u}{r} \, dr + \int_0^1 D_C \frac{u}{r} \, dr
\]

\[
\tilde{\sigma}_{2\alpha} \Delta r_{21} = B_{21} \Delta u_{21} + D_{21} \Delta u_{21} \ln \left( \frac{r_1}{r_2} \right)
- B_{21} \frac{\bar{u}_{21} T_{21} \Delta r_{21}}{\bar{T}_{21} \Delta r_{21}} - D_{21} \frac{\bar{u}_{21} T_{21} \Delta r_{21}}{\bar{T}_{21} \Delta r_{21}}. \tag{34}
\]

\[
\int_0^1 \sigma \, dr = \int_0^1 B \frac{du}{dr} \, dr + \int_0^1 D_C \frac{u}{r} \, dr - \int_0^1 B \frac{u}{r} \, dr + \int_0^1 D_C \frac{u}{r} \, dr
\]

\[
\tilde{\sigma}_{10} \Delta r_{10} = B_{10} \Delta u_{10} + D_{10} \Delta u_{10} \ln \left( \frac{r_1}{r_2} \right)
- B_{10} \frac{\bar{u}_{10} T_{10} \Delta r_{10}}{\bar{T}_{10} \Delta r_{10}} - D_{10} \frac{\bar{u}_{10} T_{10} \Delta r_{10}}{\bar{T}_{10} \Delta r_{10}}. \tag{35}
\]

At this point it is possible to develop an expression for \(\sigma_{11}\) in the [1–2] interval. With the following substitutions
\[
\begin{align*}
\beta_1 &= (tr)_1, \quad (36) \\
\beta_2 &= (tr)_2. \quad (37)
\end{align*}
\]

Eq. (32) can be written as
\[
\beta_1 \sigma_{12} - \beta_1 \sigma_{11} = \xi_{21} \sigma_{21} \Delta r_{21} - \xi_{21} r \rho \rho \left( \frac{r_1^2 - r_2^2}{3} \right). \tag{38}
\]

Next, substitute Eq. (34) into Eq. (38)
\[
\beta_2 \sigma_{21} - \beta_1 \sigma_{11} = \xi_{21} \frac{B_{21} \Delta u_{21} + D_{21} \Delta u_{21} \ln \left( \frac{r_1}{r_2} \right)}{r_2} + \xi_{21} \left[ -B_{21} \frac{\bar{u}_{21} T_{21} \Delta r_{21}}{\bar{T}_{21} \Delta r_{21}} - D_{21} \frac{\bar{u}_{21} T_{21} \Delta r_{21}}{\bar{T}_{21} \Delta r_{21}} - r \rho \rho \left( \frac{r_1^2 - r_2^2}{3} \right) \right]. \tag{39}
\]

Subtracting Eq. (39) from Eq. (30), making sure to cancel the \((\beta_2 \sigma_{21})\) term, yields the following expression for \(\sigma_{11}\) in terms of known quantities in the [1–2] interval
\[
\sigma_{11} = \left( \frac{2 \beta_2}{\beta_1 + \beta_2} \right) \frac{B_{21} \Delta u_{21} + D_{21} \Delta u_{21} \ln \left( \frac{r_1}{r_2} \right)}{r_2} \left[ -B_{21} \frac{\bar{u}_{21} T_{21} \Delta r_{21}}{\bar{T}_{21} \Delta r_{21}} - D_{21} \frac{\bar{u}_{21} T_{21} \Delta r_{21}}{\bar{T}_{21} \Delta r_{21}} - r \rho \rho \left( \frac{r_1^2 - r_2^2}{3} \right) \right]. \tag{40}
\]

A similar procedure is used to solve for \(\sigma_{11}\) over interval [0–1]. With the following substitutions
\[
\begin{align*}
\rho_0 &= (tr)_{10}, \quad (41) \\
\beta_1 &= (tr)_1. \quad (42)
\end{align*}
\]

Eq. (33) can be written as
\[
\beta_1 \sigma_{10} - \beta_2 \sigma_{10} = \xi_{10} \sigma_{10} \Delta r_{10} - \xi_{10} r \rho \rho \left( \frac{r_1^2 - r_2^2}{3} \right). \tag{43}
\]

Next, substitute Eq. (35) into Eq. (43)
\[
\beta_1 \sigma_{10} - \beta_2 \sigma_{10} = \xi_{10} \left[ B_{10} \Delta u_{10} + D_{10} \Delta u_{10} \ln \left( \frac{r_1}{r_2} \right) \right] + \xi_{10} \left[ -B_{10} \frac{\bar{u}_{10} T_{10} \Delta r_{10}}{\bar{T}_{10} \Delta r_{10}} - D_{10} \frac{\bar{u}_{10} T_{10} \Delta r_{10}}{\bar{T}_{10} \Delta r_{10}} - r \rho \rho \left( \frac{r_1^2 - r_2^2}{3} \right) \right]. \tag{44}
\]

Adding Eq. (44) to Eq. (31), making sure to cancel the \((\beta_2 \sigma_{10})\) term, yields the following expression for \(\sigma_{11}\) in terms of known quantities in the [0–1] interval
\[
\sigma_{11} = \left( \frac{2 \beta_2}{\beta_1 + \beta_2} \right) \frac{B_{10} \Delta u_{10} + D_{10} \Delta u_{10} \ln \left( \frac{r_1}{r_2} \right)}{r_2} \left[ -B_{10} \frac{\bar{u}_{10} T_{10} \Delta r_{10}}{\bar{T}_{10} \Delta r_{10}} - D_{10} \frac{\bar{u}_{10} T_{10} \Delta r_{10}}{\bar{T}_{10} \Delta r_{10}} - r \rho \rho \left( \frac{r_1^2 - r_2^2}{3} \right) \right] \tag{45}
\]

At this point two separate expressions have been found for \(\sigma_{11}\). Equate (45) and (40), substituting \(u_{11}, u_1, u_{11}\) for \(\Delta u\) and \(u\). Simplification of the this expression yields three coefficient terms for \(u_{11}, u_1, u_{11}\). The remaining terms can be thought of as being the right hand side (RHS) of the equation. The three coefficient expressions and the RHS lead to a system of \((n - 2)\) equations which solve for \(n\) radial displacement values. The remaining two equations come from discretization of the boundary conditions.

\[
\text{Coeff. } u_{11-1} = \frac{2 \beta_0}{(\beta_0 + \beta_1)} \left( \frac{A_{21}}{\Delta r_{10}} - \frac{B_{21}}{\bar{T}_{21}} \right) + \frac{\xi_{10}}{\beta_0 + \beta_1} \left( \frac{B_{21}}{\bar{T}_{21}} \right) \tag{46}
\]

\[
\text{Coeff. } u_1 = \frac{2 \beta_0}{(\beta_2 + \beta_1)} \left( -\frac{A_{21}}{\Delta r_{21}} + \frac{B_{21}}{\bar{T}_{21}} \right) + \frac{\xi_{21}}{(\beta_0 + \beta_1)} \left( \frac{B_{21}}{\bar{T}_{21}} \right) \tag{47}
\]

\[
\text{Coeff. } u_{11+1} = \frac{2 \beta_0}{(\beta_2 + \beta_1)} \left( -\frac{A_{21}}{\Delta r_{10}} + \frac{B_{21}}{\bar{T}_{10}} \right) - \frac{\xi_{10}}{(\beta_0 + \beta_1)} \left( \frac{B_{21}}{\bar{T}_{10}} \right) \tag{48}
\]

\[
\text{RHS} = \frac{2 \beta_2}{(\beta_2 + \beta_1)} \left( \frac{A_{21}}{\Delta r_{21}} + \frac{B_{21}}{\bar{T}_{21}} \right) - \frac{\xi_{21}}{(\beta_2 + \beta_1)} \left( \frac{B_{21}}{\bar{T}_{21}} \right) - \frac{2 \beta_0}{(\beta_0 + \beta_1)} \left( A_{10} \frac{\bar{x}_{10}}{\bar{T}_{10}} + B_{10} \frac{\bar{x}_{10}}{\bar{T}_{10}} \right) - \frac{\xi_{10}}{(\beta_0 + \beta_1)} \left( \frac{B_{21}}{\bar{T}_{10}} \right) - \frac{\xi_{10}}{(\beta_0 + \beta_1)} \left( \frac{B_{21}}{\bar{T}_{10}} \right) \left( \frac{r_2^2 - r_1^2}{3} \right) \tag{49}
\]

A3. Boundary conditions

The discretization of the first boundary condition is shown below. A three point forward difference scheme is applied to Eq. (26). This method is accurate enough to maintain system wide second order convergence. Reorganization of the terms yields three displacement coefficients and the equation right hand side. As discussed before, for a standard rim loaded disk \(\sigma_i\) at boundary condition #1 will be zero and the equation right hand side will consist only of thermal stress terms.
\[ \sigma_r = - \frac{n_b m_b r_0}{2 \pi r_1} \frac{\partial^2}{\partial \theta^2} + A_0 \frac{4u_2 - u_3 - 3u_1}{r_3 - r_1} + B_1 \frac{u_1}{r_1} \]
\[ - A_1 x_1 T_1 - B_1 x_0 T_1. \]  
\[ \text{Coef.: } u_1 = \frac{B_1}{r_1} - \frac{3A_1}{r_3 - r_1}. \]  
\[ \text{Coef.: } u_2 = \frac{4A_1}{r_3 - r_1}. \]  
\[ \text{Coef.: } u_3 = - \frac{A_1}{r_3 - r_1}. \]  
\[ \text{RHS } = - \frac{n_b m_b r_0}{2 \pi r_1} \frac{\partial^2}{\partial \theta^2} + A_1 x_1 T_1 + B_1 x_0 T_1. \]

For the second boundary condition a similar procedure is followed. A three point backward difference scheme is applied to Eq. (27). Reorganization of the terms yields an additional three displacement coefficients and the equation right hand side. For a rim loaded disk \( \sigma_r \), at boundary condition #2 will have a known positive value.

\[ \sigma_r = - \frac{n_b m_b r_0}{2 \pi r_1} \frac{\partial^2}{\partial \theta^2} + A_0 \frac{3u_n - 4u_{n-1} + u_{n-2}}{r_n - r_{n-2}} + B_n \frac{u_n}{r_n} \]
\[ - A_n x_n T_n - B_n x_0 T_n. \]  
\[ \text{Coef.: } u_{n,2} = \frac{A_n}{r_n - r_{n-2}}. \]  
\[ \text{Coef.: } u_{n,1} = -4A_n \frac{r_n - r_{n-2}}{r_n - r_{n-2}}. \]  
\[ \text{Coef.: } u_{n} = \frac{B_n}{r_n} + \frac{3A_n}{r_n - r_{n-2}}. \]  
\[ \text{RHS } = \frac{n_b m_b r_0}{2 \pi r_1} \frac{\partial^2}{\partial \theta^2} + A_n x_n T_n + B_n x_0 T_n. \]

A.4. Final matrix equation

The discretization shown in the previous section yields a primarily tridiagonal matrix equation with the radial displacement \( u \) as the unknown variable. The numbers in the matrices represent the equation numbers derived previously. Elementary row operations with rows \( 1, 2, (n-1), \) and \( n \) are used to eliminate the extra terms in the first and last row, leaving a truly tridiagonal matrix. Solution of this system is then carried out using the Thomas algorithm.

\[
\begin{pmatrix}
(51) & (52) & (53) & 0 & 0 & 0 & 0 \\
(46) & (47) & (48) & 0 & 0 & 0 & 0 \\
0 & (46) & (47) & (48) & 0 & 0 & 0 \\
0 & 0 & 0 & (46) & (47) & (48) & 0 \\
0 & 0 & 0 & 0 & (46) & (47) & (48) \\
0 & 0 & 0 & 0 & 0 & (56) & (57) & (58) \\

\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_n-2 \\
u_n-1 \\
u_n
\end{pmatrix}
\]

Appendix B. Advantages of the T-Axi Disk stress model

B.1. Alternative discretization method

The method shown has a distinct advantage over other, simpler methods. The results of a study comparing the T-Axi Disk discretization to a simple second order accurate finite difference scheme are shown. The test cases for this comparison do not include any thermal loading, so the thermal stress term has been omitted from the simplified stress equations. The equations are also simplified by assuming isotropic materials.

**Equilibrium:**

\[
\frac{d}{dr} (\tau \sigma_r) - \tau \sigma_r = -\rho \omega^2 r^2.
\]

**Radial stress:**

\[
\sigma_r = \frac{E}{1 - \nu^2} \left[ \frac{du}{dr} + \frac{v u}{r} \right]
\]

**Tangential stress:**

\[
\sigma_\theta = \frac{E}{1 - \nu^2} \left[ \frac{u}{r} + \frac{du}{dr} \right]
\]

**Combined governing equations:**

Combine the three equations above, leading to the following second order governing equation in terms of the radial displacement:

\[
\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \frac{u}{r} = \frac{\rho \omega^2 r}{E} (v^2 - 1).
\]

Eqs. (61)–(64) show the governing equations for an arbitrary thickness disk. Eq. (64) is discretized using a second order standard central difference approach. Three point forward and backward schemes are used at the boundary conditions. The resulting matrix equation is tridiagonal except for the first and last equations and is solved using the same modified Thomas method used in T-Axi Disk.

B.2. Comparison results

For a smooth thickness distribution discretized with a uniformly spaced grid, the T-Axi Disk discretization is similar to a common second order accurate central difference method. Fig. 12 shows the Von Mises stress results from T-Axi Disk compared to results using the standard central difference method. The disk analyzed in this example has a smooth profile without any discontinuities in thickness, discretized with an evenly spaced grid. The Von Mises stress results from an axisymmetric ANSYS finite element analysis are included for comparison. For this simple case it is easy
to see that both a simple central difference and the T-Axi Disk method result in well converged solutions with 100 pts.

However, some common disk parameterizations contain thickness discontinuities. To accurately model these discontinuities the grid can no longer be evenly spaced, which makes the central difference method less effective. Even worse, a central difference method inevitably leads to a case where the discretization stencil straddles the discontinuity point, effectively rounding off the sharp corner and throwing off the solution everywhere in the continuum. The number of points needed for convergence is therefore much higher. Fig. 13 shows the Von Mises stress results for the optimized $E^3$ third stage web disk. The results clearly show that a central difference approach will require thousands of points to achieve a satisfactory level of convergence. The method implemented in T-Axi Disk is much more effective. Integration of the equilibrium equation instead of differentiation, as shown in Eqs. (32) and (33), allows the system to deal with thickness discontinuities without needing a large number of grid points. Although not shown in this example, the discretization method in T-Axi Disk also allows discontinuities in material properties, something not possible with the central difference approach.

References

[23] Turner MG, Merchant A, Bruna D, A turbomachinery design tool for teaching design concepts for axial-flow fans, compressors, and turbines, Barcelona, Spain; 2006 [ASME paper number GT2006-90105].