Homework 4
Partial Differential Equations: Analytical Methods

1

A cylinder (radius $R = 1$, height $Z = 1$) is at a ground potential $u = 0$ at its base and over its side surface while the top surface is maintained at a constant potential $u = 1$. Use the method of separation of variables to obtain the potential inside the cylinder under steady state conditions. Use the separation of variables method to determine the function $u(r, z)$ that satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{\partial^2 u}{\partial z^2} = 0$$

subject to

$$\begin{aligned} u(0, r) &= 0 \\ u(1, z) &= 0 \\ u(r, 1) &= 1 \end{aligned}$$

Note that, because of the symmetry, $\partial u/\partial r = 0$ at $r = 0$ for all values of z.

2

A long metal rod ($X = 1$) is embedded in an highly insulating medium. Initially, its temperature is $u = 1$ along its entire length. At time $t = 0$ the right end is instantaneously brought to a temperature of $u = 0$ while the left end is maintained at $u = 1$. Use the separation of variables method to determine the function $u(x, t)$ that satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to

$$u(0,t) = 1$$
$$u(1,t) = 0$$
$$u(x,0) = 1$$

3

An taut elastic string ($X = 1$) is firmly held at its ends while deflected at its midpoint by an amount $u_{max} = 0.1$. The string is then released and allowed to perform undamped oscillations about its equilibrium position. Use the separation of variables method to determine the function $u(x,t)$ that satisfies

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

subject to

$$u(0,t) = 0$$
$$u(1,t) = 0$$

and

$$u(x,0) = \begin{cases} x/5 & (x < 0.5) \\ (1-x)/5 & (x > 0.5) \end{cases}$$

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