Conduction Heat Transfer
Solved Sample Problems

1

The temperature T is maintained at 0 at three edges of a square plate and at 100 at the fourth edge. The plate is 100 cm in side. For steady state conditions

a) Find $T(x, y)$ in the plate.

b) Calculate $T(50, 50)$.

Answer:

a) Assume $T_p = X(x)Y(y)$. Substituting into Laplace's equation yields

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = k^2$$

with general solutions

$$X(x) = a\sin kx + b\cos kx$$

$$Y(y) = \alpha e^{ky} + \beta e^{-ky}$$

Using the three homogeneous boundary conditions reduces the above to

$$X_n(x) = a_n \sin(\frac{n\pi}{100}x)$$

and

$$Y_n(y) = \alpha_n(e^{ky} - e^{-ky}) = \alpha_n \sinh(\frac{n\pi}{100}y)$$

Now, substituting $T_p = X(x)Y(y)$ and superimposing for all n yields

$$T(x, y) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{100}x)\sinh(\frac{n\pi}{100}y)$$

Finally, introducing the initial condition $T(x,100) = 100$ gives

$$100 = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{100}x)\sinh(n\pi) = \sum_{n=1}^{\infty}[A_n \sinh(n\pi)]\sin(\frac{n\pi}{100}x) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{100}x)$$

which is the Fourier sine series representation of 100 with the Fourier coefficients

$$c_n = A_n \sinh(n\pi) = \frac{2}{100}\int_0^{100} 100\sin(\frac{n\pi}{100}x)dx = 2\frac{100}{n\pi}[1-\cos(n\pi)]$$

I.e.

$$A_n = \begin{cases} 0 & n \ \ even \\ \frac{400}{n\pi \sinh n\pi} & n \ \ odd \end{cases}$$

With the above the desired solution can now be rearranged to

$$T(x,y) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{100}x)\frac{\sinh(\frac{n\pi}{100}y)}{\sinh(n\pi)}$$

with

$$c_n = \begin{cases} 0 & n \ \ even \\ \frac{400}{n\pi} & n \ \ odd \end{cases}$$

b) To find $T(50,50)$ use the first four terms in the above series to get

$$T(x,y) = \frac{400}{\pi}\sin(\frac{\pi}{100}50)\frac{\sinh(\frac{\pi}{100}50)}{\sinh(\pi)} + 0 + \frac{400}{3\pi}\sin(\frac{3\pi}{100}50)\frac{\sinh(\frac{3\pi}{100}50)}{\sinh(3\pi)} + 0 =$$
$$= 25.37 - 0.3812 = 24.98$$

2

The temperatures at the ends $x = 0$ and $x = L = 100$ of a 100 cm long rod with insulated sides are held at temperatures of 0 and 100 , respectively until reaching steady state. Then, the temperatures at the ends are interchanged. Find $T(x,t)$.

Answer: The solution to the problem of $T = T_1$ at $x = 0$ and $T = T_2$ at $x = 100$, is

$$T(x,t) = T_1 + (T_2 - T_1)\frac{x}{l}\sum_{n=1}^{\infty} a_n \sin(\frac{n\pi}{l}x)e^{-\frac{n^2\pi^2\alpha}{l^2}t}$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx + \frac{2}{n\pi}(T_2\cos n\pi - T_1)$$

This can be applied directly noticing that here $f(x) = 100x/L$ is the initial condition (obtained from the previous steady state) and that $T_1 = 100$ and $T_2 = 0$. The Fourier coefficients are then

$$a_n = \frac{1}{50}[-\frac{100^2}{n\pi}(-1)^n] - \frac{200}{n\pi} = \frac{200}{n\pi}[(-1)^{n+1} - 1] = \begin{cases} -400/n\pi & even \\ 0 \quad odd \end{cases}$$

The result is

$$\begin{aligned} T(x,t) &= 100 - x - \sum_{n\geq 1, even} \frac{400}{n\pi}\sin(\frac{n\pi}{100}x)e^{-\frac{n^2\pi^2\alpha}{100^2}t} = \\ &= 100 - x - \sum_{m=1} \frac{400}{2m\pi}\sin(\frac{2m\pi}{100}x)e^{-\frac{(2m)^2\pi^2\alpha}{100^2}t} = \\ &= 100 - x - \frac{200}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin(\frac{m\pi}{50}x)e^{-\frac{m^2\pi^2\alpha}{2500}t} \end{aligned}$$

3

A large, 0.1 meter thick steel slab emerges from a rolling mill with a uniform temperature of 1000 degrees Celsius and is left to cool by convection into an environment at zero degrees Celsius. For this steel $k = 50W/mK$, $\rho = 7,900kg/m^3$ and $C_p = 470J/kgK$. Assume the heat transfer coefficient is $h = 250W/m^2K$.

Evaluate the first three terms in the series solution of this problem obtained by separation of variables when the temperature at the center of the slab is 500 degrees Celsius.

Answer:

Assume heat is convected aways from both sides of the slab equally. This allows us do the analysis on just one half of the slab (i.e. $L = 0.05$). The mathematical formulation of the problem is to find $T(x,t)$ such that

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

where $\alpha = k/\rho C_p = 50/(7,900 \times 470) = 1.34 \times 10^{-5}$.

The boundary conditions consist of the symmetry condition at $x = 0$

$$-k\frac{\partial T}{\partial x} = \frac{\partial T}{\partial x} = 0$$

and the convective loss into an environment at $T_\infty = 0$ at $x = L = 0.05$,

$$k\frac{\partial T}{\partial x} + hT = 50\frac{\partial T}{\partial x} + 250T = 0$$

The initial condition is

$$T(x, 0) = 1000$$

for all x when $t = 0$.

Assume the solution is of the form $T(x, t) = X(x)\Gamma(t)$ and substitute in the heat equation to obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt}$$

The left hand side is only a function of x. The right hand side is only a function of t. Since x and t are independent of each other the equality is only possible if both terms are equal to a constant.

Hot objects exposed to cold environments cool down and cold objects exposed to warm environments heat up. This requires the constant above to be a negative real number. To make sure of this we call it $-\beta^2$.

Therefore, we obtain the following two equations, first

$$\frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt} = -\beta^2$$

for which a general solution is readily obtained by integration as (check!)

$$\Gamma(t) = Ce^{-\beta^2\alpha t}$$

where C is a constant of integration and

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\beta^2$$

for which a general solution is obtained by inspection as (check!)

$$X(x) = A\cos(\beta x) + B\sin(\beta x)$$

However, at $x = 0$ we require (check!)

$$\frac{\partial T}{\partial x} = \frac{\partial(X\Gamma)}{\partial x} = \Gamma\frac{dX}{dx} = \frac{dX}{dx} = 0 = -A\beta\sin(\beta x) + B\beta\cos(\beta x)$$

so that, necessarily $B = 0$ and $X(x) = A\cos(\beta x)$ (check!).

Furthermore, at $x = L$, (check!)

$$\frac{dX}{dx} + \frac{h}{k}X = 0 = -\beta\sin(\beta L) + \frac{h}{k}\cos(\beta L)$$

For this to be true, the values of β must satisfy the equation (check!)

$$(\beta_n L)\tan(\beta_n L) = \frac{hL}{k}$$

This equation as infinitely many roots β_n for $n = 1, 2, ...$ The roots are the eigenvalues of this problem. The roots can readily be obtained either graphically or numerically. In our problem, $hL/k = 250(0.05)/50 = 0.25$ and the above equation becomes

$$z_n\tan(z_n) = 0.25$$

with $z_n = 0.05 \times \beta_n$. The first four roots are then (check!) $z_1 \approx 0.48$, $z_2 \approx 3.22$ and $z_3 \approx 6.325$ so that $\beta_1 \approx z_1/L = 0.48/0.05 = 9.6$, $\beta_2 \approx 64.4$ and $\beta_3 \approx 126.5$. The associated eigenfunctions of the problem are then

$$X_n(x) = X(\beta_n, x) = A_n\cos(\beta_n x)$$

The particular solution obtained by separation of variables and satisfying the heat equation as well as the boundary conditions is then

$$T_n(x,t) = X(\beta_n, x)\Gamma(\beta_n, t) = c_n\cos(\beta x)e^{-\alpha\beta_n^2 t}$$

where $c_n = A_n C$. A more general solution is obtained by linear combination of the above solutions, i.e.

$$T(x,t) = \sum_{n=1}^{\infty} T_n(x,t) = \sum_{n=1}^{\infty} c_n\cos(\beta_n x)e^{-\alpha\beta_n^2 t}$$

All is left to do is introduce the initial condition, i.e. at $t = 0$,

$$T_i = 1000 = \sum_{n=1}^{\infty} c_n\cos(\beta_n x)$$

But this is just the Fourier cosine series representation of the number 1000. Now multiply both sides of the above by $\cos(\beta_m x)$ and integrate from 0 to L

$$\int_0^L T_i\cos(\beta_m x)dx = \int_0^L c_n\cos(\beta_n x)\cos(\beta_m x)dx$$

But since the cos functions constitute an orthogonal set, the right hand side becomes

$$\int_0^L \sum_{n=1}^{\infty} c_n \cos(\beta_n x)\cos(\beta_m x)dx = \sum_{n=1}^{\infty} \int_0^L c_n \cos(\beta_n x)\cos(\beta_m x)dx =$$

$$= \int_0^L c_n \cos(\beta_n x)^2 dx = \frac{1}{2\beta_n}(\beta_n L + \sin(\beta_n L)\cos(\beta_n L))$$

Therefore (check!)

$$c_n = \frac{2T_i \sin(\beta_n L)}{\beta_n L + \sin(\beta_n L)\cos(\beta_n L)}$$

Finally, the desired solution is then

$$T(x,t) = 2T_i \sum_{n=1}^{\infty} \frac{\sin(\beta_n L)\cos(\beta_n x)}{\beta_n L + \sin(\beta_n L)\cos(\beta_n L)} e^{-\alpha\beta_n^2 t}$$

Using just the first term of the series, introducing numerical values and focusing on the temperature at $x = 0$ yields (check!)

$$T(x,t) \approx 2000 \frac{\sin(0.48)}{0.48 + \sin(0.48)\cos(0.48)} \exp(-1.34 \times 10^{-5}(9.6)^2 t) = 1038 e^{-1.23\times 10^{-3} t}$$

The following table gives numerical values obtained from the above equation for a few values of time (check!)

time (s)	Temperature (C)
600	496
1200	237
1800	113

4

Find the function $T(x,t)$ in $0 \leq x < \infty$ satisfying

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

and subject to

$$T(0,t) = f(t)$$

$$T(x \to \infty, t) = 0$$

and

$$T(x,0) = 0$$

using the Laplace Transform method.

Answer:

Taking Laplace transforms one gets

$$\frac{d^2\bar{T}(x,s)}{dx^2} - \frac{s}{\alpha}\bar{T}(x,s) = 0$$

$$\bar{T}(0,s) = \bar{f}(s)$$

$$\bar{T}(x \to \infty, s) = 0$$

The solution of this problem is

$$\bar{T}(x,s) = \bar{f}(s)e^{-x\sqrt{s/\alpha}} = \bar{f}(s)\bar{g}(x,s) = \mathcal{L}[f(t) * g(x,t)]$$

Inversion then produces

$$T(x,t) = f(t) * g(x,t) = \int_0^t f(\tau)g(x,t-\tau)d\tau$$

Inversion of $\bar{g}(x,s)$ to get $g(x,t)$ finally gives

$$T(x,t) = \frac{x}{\sqrt{4\pi\alpha}} \int_{\tau=0}^{t} \frac{f(\tau)}{(t-\tau)^{3/2}} \exp[-\frac{x^2}{4\alpha(t-\tau)}]d\tau$$

If $f(t) = T_0 =$ constant, the solution is

$$\bar{T}(x,s) = \frac{T_0}{s}\exp(-x\sqrt{s/\alpha})$$

and inverting the transform yields

$$T(x,t) = T_0 erfc(\frac{x}{2\sqrt{\alpha t}})$$

where the **complementary error function** $erfc(z)$ is given by

$$erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} \exp(-z'^2)dz'$$

5

Find a bounded solution $u(r,t)$ of the following problem using the Laplace transform method.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}$$

subject to

$$u(r,0) = 0$$

and

$$u(a,t) = u_0$$

where $t > 0$, $0 < r < a$ with $a =$ constant.

Answer:
Taking Laplace transforms of the terms in the given PDE yields

$$\frac{d^2\bar{u}(r,s)}{dr^2} + \frac{1}{r}\frac{d\bar{u}(r,s)}{dr} = s\bar{u}(r,s)$$

This is a Bessel-type equation with general solution given by

$$\bar{u}(r,s) = AJ_0(ir\sqrt{s}) + BY_0(ir\sqrt{s})$$

Since we look for a bounded solution, necessarily $B = 0$. Now transformation of the boundary condition at $r = a$ yields

$$\bar{u}(a,s) = \frac{u_0}{s}$$

and combining with the above gives

$$A = \frac{u_0}{s}\frac{1}{J_0(ia\sqrt{s})}$$

so that the solution for $\bar{u}(r,s)$ becomes

$$\bar{u}(r,s) = u_0\frac{J_0(ir\sqrt{s})}{sJ_0(ia\sqrt{s})}$$

Finally, the inverse transformation gives

$$u(r,t) = u_0[1 - 2\sum_{n=1}^{\infty}\frac{e^{-\lambda_n^2 t/a^2}J_0(\lambda_n r/a)}{\lambda_n J_1(\lambda_n)}]$$

where λ_n are the roots of

$$J_0(\lambda_n) = 0$$