Numerical calculation of wear in mechanical systems

Kurt Frischmuth a, Dirk Langemann b,*

a University of Rostock, Institute of Mathematics, Ulmenstraße 69, D-18051 Rostock, Germany
b Technical University of Braunschweig, Institute of Computational Mathematics, Pockelsstr. 14, D-38106 Braunschweig, Germany

Received 2 August 2010; received in revised form 19 January 2011; accepted 16 May 2011
Available online 26 May 2011

Abstract

The simulation of wear processes is dominated by the feedback between system dynamics and wear, which have completely different characteristic time scales. We present a method for decoupling the time scales. Its central ideas are the introduction of a wear density on the entire wearing surface and the establishment of an evolution equation in the slow time scale. The sample problem of a wearing disk rolling on a rigid rail is studied. In particular, we discuss the non-monotonous influence of the speed on the wear evolution and on the polygonalization of wheels.

© 2011 IMACS. Published by Elsevier B.V. All rights reserved.

MSC: 65Z05; 70E55; 37M05

Keywords: Numerical wear simulation; Polygonalization of railway wheels; Multiple time scales

1. Introduction

Wear processes are governed by a feedback between the dynamics of a mechanical system and the evolution of a wearing surface which recurrently comes into contact with another surface, which may be subject to wear as well or may remain unchanged. Important components of wear models are contact mechanics [13,15,16] including friction and a wear law [2,3]. The feedback loop of wear processes is depicted in Fig. 1.

The wear of railway wheels is a central application of simulating wear evolution processes [18–20,24]. The simulations are accompanied by experiments [6] and measurements [14]. Numerical simulations of wear processes – and in particular of the polygonalization of railway wheels – are found in [5] by harmonic analysis, in [19] for an elastic wheelset, the inner motion of which is described by a series of eigenscillations, in [24] focusing on the lateral wear of profiles of railway wheels and in [20,21] dealing with numerical techniques. Further numerical approaches are found in [1].

The characteristic time scales of system dynamics and wear are very different. For instance, a railway wheel of a high speed train at 200 km/h turns about 17 times per second, and after about 150,000 km of traveling or fifty million of wheel revolutions, which means about 2 months in a regular railway service, the wheel has lost a material layer of about 150 μm thickness [19,21,24].
Fig. 1. Feedback loop between dynamics and wearing geometry. The dissipated friction power density, as a mechanical quantity, effects wear, i.e. a change of the geometry. The geometry influences the dynamical behavior via the contact conditions. The time scales are dramatically different in realistic applications. While the characteristic scale of the dynamic is in the range of 100 milliseconds, the slow time scale of the wear has a characteristic range of hours or days.

The present paper continues investigations from [10,11] aiming in the separation of the characteristic time scales. Our method to decouple the time scales can be regarded as a specification of the general heterogeneous multiscale method from [8] for the feedback problem between system dynamics and wear. But the focus of our investigations lies on a meaningful formulation of the wear evolution by introducing a wear density rather than on a purely numerical procedure.

Now, we give a mathematical formulation of the coupled process of system dynamics and wear in Section 2, which focuses on the feedback between system dynamics and wear. We refer to [9] for a rigorous approach in continuum mechanics. We present different simulation techniques in Section 3, in particular a new method to decouple the time scales. Then, we discuss the wear evolution of a disk wheel model, given in Section 4, focusing on the polygonalization of railway wheels in dependence of the traveling speed. These simulation results are calculated by a new technique to decouple the different time scales. The results are given in Section 5 and compared with the results of the traditional handling with an amplified wear coefficient in Section 6.

2. Basic elements in wear simulation

2.1. Wearing surface

We describe the wearing body as a local and temporal continuum. The surface, which evolves under the influence of wear, is denoted by \( \mathcal{U} = \mathcal{U}(t) \subset \mathbb{R}^3 \) for the time \( t \in [0, t_{\text{fin}}] \). We use \( \mathcal{U}_0 = \mathcal{U}(0) \) as reference surface and parameterize its particles by \( p(\sigma) \in \mathbb{R}^3 \), i.e. \( \mathcal{U}_0 = \{ p(\sigma), \sigma \in S \subset \mathbb{R}^2 \} \). We assume that \( \mathcal{U}(t), t \in [0, t_{\text{fin}}] \) can be diffeomorphically mapped to \( \mathcal{U}_0 \) by a projection in normal direction \( n = n(\sigma) \) to \( \mathcal{U}_0 \), and we get

\[
\mathcal{U}(t) = \{ p(\sigma) + u(t, \sigma)n(\sigma), \sigma \in S \}, t \in [0, t_{\text{fin}}]
\]

with a sufficiently smooth function \( u : [0, t_{\text{fin}}] \times S \to \mathbb{R} \) describing the wearing surface. The smoothness of \( u \) and \( \mathcal{U}_0 \) implies smooth manifolds \( \mathcal{U}(t) \) at every time instant \( t \in [0, t_{\text{fin}}] \) and a smooth evolution \( p(\sigma) + u(t, \sigma)n(\sigma) \) at each parameter \( \sigma \in S \). The normal directions are identified due to smallness of \( u \) itself and its curvature. For a more general approach we refer to [22].

We have \( u(0, \sigma) = 0 \). The value \( u(t, \sigma) \leq 0 \) is monotonously decreasing in \( t \) for every parameter \( \sigma \in S \). Now, the wear speed is \( \dot{u}(t, \sigma) = u(t, \sigma) \leq 0 \). The accumulated material loss at the point \( p(\sigma) \) until the time instant \( t \) is \( -u(t, \sigma) \). The function \( u(t, \cdot) : S \to \mathbb{R} \) is called the wear state of the evolving surface, the geometry of the wearing system, or simply the shape. The set of all admissible wear states is denoted by \( \mathcal{U} \). It is called the shape space. Model idealizations can lead to one-dimensional manifolds \( \mathcal{U}(t) \subset \mathbb{R}^2 \) with \( \sigma \in S \subset \mathbb{R} \), cf. the disk wheel in Section 4, or even to dimensionless \( \mathcal{U}(t) \in \mathbb{R} \).

2.2. Mechanical system

The short-time dynamics of the mechanical system with \( b \) degrees of freedom is described by the time dependence of the mechanical state \( y(t) = (q(t), \dot{q}(t))^T \in \mathbb{R}^{2b} \) with the position vector \( q = q(t) \in \mathbb{R}^b \). The mechanical state space is denoted by \( \mathcal{Y} = \mathbb{R}^{2b} \). The equation of motion reads

\[
\ddot{q}(t) = F(t, q(t), \dot{q}(t); u(t, \cdot)) + F_{\text{ext}}(t),
\]
with the mass tensor $\bar{m}$ and the deterministic force $F$ depending on the position $q$, on the velocity $\dot{q}$, on the time $t$ by a possible external control and on the shape $u$.

The dependence of the force $F$ on the shape $u$ is imposed by contact conditions between the evolving surface $\partial u(t)$ and other bodies within the considered mechanical system. The contact conditions are modeled by stiff contact springs [13]. This can be regarded as a compromise between a numerically very expensive fully elastic model, which would provide a realistic contact area, and a rigid-body model, which would result in point contact and in a differential algebraic system instead of Eq. (2). Due to the contact spring, Eq. (2) is a stiff ordinary differential equation, which can be solved by implicit standard methods [12].

Furthermore Eq. (2) contains an external force term $F_{\text{ext}}(t)$ containing additional external forces, which behave non-deterministically on time scales relevant for wear. Under the assumption that the external control and the non-deterministic external force are continuous, the position vector $q(t)$ is twice differentiable.

We combine the evolution of the mechanical state $y(t)$ in the evolution equation

$$\dot{y} = f(t, y, u) \quad \text{with} \quad f : [0, t_{\text{fin}}] \times Y \times U \rightarrow Y.$$  

The particles of the wearing surface, which are in contact to contact partners, depend on the mechanical state $y = y(t) \in Y$.

Let $\vartheta = \vartheta(y) \in S$ be the parameter of a particle in contact, i.e. $\vartheta$ denotes a point of $\partial u(t)$ in the parameterization in Eq. (1) which presently touches a contact partner. The contact spring produces a normal force $F_N(t, \vartheta(y(t)); y, u)$ acting on the particle $p(\vartheta) + u(t, \vartheta)n(\vartheta) \in \partial u(t)$.

The relative velocity between the contact partners is denoted by $s = s(t, \vartheta)$. A friction law determines the tangential force $F_T = F_T(t, \vartheta(y); y, u)$. In the case of rolling friction, the tangential force $F_T$ is proportional to the normal force $F_N$ and to the relative velocity $s$ [18]. If the effects in the contact set are handled in a local resolution, Coulomb’s law is necessary [4,15,17].

The contact conditions pose a central difficulty, because using a rigid-body system with a finite number of degrees of freedom is usual and necessary for any numerical simulation of the longterm dynamics [23]. But, the contact set between two rigid bodies typically consists of a single point or of several separated points. Remark 1 shows that a contact set of a dimension lower than the dimension of the wearing surface is contradicting to the smoothness condition. Therefore, it is desirable to deal with force densities, and the elastic behavior of the material has to be computed or at least approximated in the neighborhood of the contact [16,17].

The force densities in the contact area can be approximated by a suitable footprint $\chi \in C^2_{\text{loc}}(S)$ with $\chi(\sigma) \geq 0$ for $\sigma \in S$ and with the volume

$$\int_S \chi(\sigma) \, d\sigma(\sigma) = 1,$$

which smooths the concentrated normal force to a normal force density

$$f_N(t, \sigma) = \chi(\sigma - \vartheta)F_N(t, \vartheta)$$

and the tangential force to the density $f_T(t, \sigma) = \chi(\sigma - \vartheta)F_T(t, \vartheta)$, respectively. The support of the footprint $\text{supp} \chi \subset S$ gives an approximation of the elastic contact area.

### 2.3. Wear law

A wear law maps a couple $(y, u)$ of a mechanical state $y$ and a shape $u$ to the wear speed, i.e. to the evolution speed of the wearing surface. The wear law is an assignment

$$g : Y \times U \rightarrow U \quad \text{by} \quad g : (y(t), u(t, \cdot)) \mapsto g(y(t), u(t, \cdot)) = \dot{u}(t, \cdot).$$

The most commonly used wear law is Archard’s law [2,3], which sets the wear speed proportional to the dissipated power density in the contact patch.

Let be $f_T = f_T(t, \sigma)$ the tangential force density, which vanishes outside the contact area, and $s = s(t, \sigma)$ the relative velocity, which may depend on $\sigma$ in a global or local elastic setting but which is a position independent slip vector in a
simple rigid-body model. Now, Archard’s law claims the proportionality between the frictional power dissipation and the wear speed

\[ \dot{u}(t, \sigma; y, u) = \beta s(t, \sigma; y, u) \cdot f_T(t, \sigma; y, u) \leq 0 \]  
(6)

with a proportionality constant \( \beta \), called wear coefficient [2,3]. Eq. (6) is nonlinear because the tangential force density \( f_T \) depends on the relative velocity \( s \), too. In the case of rolling friction, \( f_T \) is proportional to \( s \) and thus, \( \dot{u} \) is proportional to \( s^2 \). Since the relative velocity \( s \) and the tangential force \( f_T \) depend on the mechanical state \( y \) and on the shape \( u \), the nonlinear Eq. (6) specifies the wear law (5). The popularity of Archard’s law is rather caused by its simplicity than by its accuracy. Yet there is no generally accepted universal wear law.

**Remark 1.** A frictional power density, which is concentrated at a single particle on the surface, would concentrate the wear speed \( \dot{u} = g(y, u) \) to this single particle at the time instant \( t \). That means, that \( \dot{u} \) would be a real distribution, \( u \) becomes discontinuous, and any regularity would be lost.

### 3. Coupled system

The coupled system of dynamics (3) and wear evolution (5) reads

\[ \dot{y} = f(t, y, u), \quad y(0) = y_0 \]  
(7)

\[ \dot{u} = g(y, u), \quad u(0, \cdot) = u_0 \]  
(8)

for \( t \in [0, t_{\text{fin}}] \). A typical property of wear processes is the difference between the characteristic time scales of the fast dynamics and the slow wear evolution.

Furthermore, the external control and the non-deterministic force term \( F_{\text{ext}}(t) \) in Eq. (2) contained in Eq. (7) as time-dependence, describe fast changing exogenous influences. Since the geometry evolves slowly, we suppose that the wear process does not decisively depend on a particular realization of the exogenous influence or the particular initial values \( y_0 \) for the dynamics. The wear process rather depends on the distribution of the exogenous influences.

The standard approach to a system of the form (7, 8) would be numerical integration over \([0, t_{\text{fin}}]\). However, in our case an equal treatment of both equations turns out to be not suitable. In fact, the high frequency of changes in the dynamical component would enforce an unreasonably small time step on the wear evolution, while the very small rate of change in the wear equation would require an unnecessarily high number of repetitive cycles of nearly identical calculations [7].

#### 3.1. Amplified wear coefficient

A first idea and a typical traditional way [19,24] of handling the coupled system (7, 8) is the artificial adjustment of the time scales by an amplification factor \( a \gg 1 \). It exaggerates the wear speed relatively to the dynamics. We denote \( \tau = t/a \), and the coupled system (7, 8) now reads

\[ \frac{dy}{d\tau} = af(\tau, y, u), \quad y(0) = y_0, \]  
(9)

\[ \frac{du}{d\tau} = ag(y, u), \quad u(0, \cdot) = u_0. \]  
(10)

for \( \tau \in [0, t_{\text{fin}}/a] \). The exaggeration of the wear speed coincides with slowing the dynamics compared to the wear evolution by leaving out the amplification factor \( a \) in Eq. (9). The changed geometry \( w \) obeys the scaled evolution problem

\[ \frac{dy}{d\tau} = f(\tau, y, w), \quad y(0) = y_0, \]  
(11)

\[ \frac{dw}{d\tau} = ag(y, u), \quad w(0, \cdot) = u_0. \]  
(12)

The scaling reduces the numerical effort due to the shortened time interval and there is the hope that \( w(\tau, \cdot) \approx u(t, \cdot) \) with acceptable accuracy while the dynamical behavior is only slightly influenced. However, it turns out that this
amplified wear factor can result in a wear evolution dominated by the friction power density dissipated during the first rotation, see Section 6.

3.2. Decoupled time scales

First, we regard the mechanical sub-system with a frozen geometry $u$ of the surface $U$. We consider the trajectory $y = y(t)$ of the mechanical sub-system for fixed $u$ under the exogenous influence as random variable. It is the solution of Eq. (7) alone. The trajectory $y$ for $t \rightarrow \infty$ overlaps a subset of the mechanical state space. It reproduces the probabilistic distribution of the mechanical states in $Y$.

Let be $\Delta Y \subseteq Y$ a measurable subset of the mechanical state space, i.e. $\Delta Y \in B(\mathbb{R}^{2b})$. The probability $\mathcal{P}(y(t) \in \Delta Y)$ is determined by the measure $\nu_Y(\Delta Y; u)$, which of course depends on $u$. It is the time share of the trajectory being inside $\Delta Y$. With the characteristic function

$$1_{\Delta Y}(y) = \begin{cases} 1 & \text{for } y \in \Delta Y, \\ 0 & \text{for } y \notin \Delta Y, \end{cases}$$

the time share of the trajectory inside $\Delta Y$ reads

$$\nu_Y(\Delta Y; u) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T 1_{\Delta Y}(y(t)) \, dt. \quad (13)$$

Now, we consider the probability that a certain material loss happens at $\sigma \in S$. For fixed $u$, the wear law $g$ maps every mechanical state $y$ on a wear speed at every $\sigma \in S$. We restrict the map $g$ to a certain $\sigma$, and we get

$$g|_{u,\sigma} : y \mapsto \dot{u}(\sigma) \in \mathbb{R}. \quad (14)$$

Let be $\Delta R \in B(\mathbb{R})$ a measurable subset of $\mathbb{R}$. The measure $\nu_U(\Delta R; u)(\sigma)$ is the probability $\mathcal{P}(g|_{u,\sigma}(y(t)) \in \Delta R)$, i.e. $\nu_U$ measures the occurrence of a wear speed in $\Delta R$ at the parameter $\sigma$. Then, $\nu_U$ is the image measure of $\nu_Y$ under the map $g|_{u,\sigma}$. It depends on $\sigma$. The expectation of $\dot{u}$ at $\sigma$ is

$$G(u)(\sigma) = \int_{\mathbb{R}} \rho \, d\nu_U(\rho; u)(\sigma). \quad (15)$$

The mean wear speed or wear density

$$G : U \rightarrow U \quad \text{with} \quad u \mapsto G(u)$$

maps a geometry to the mean evolution speed of this geometry. If the external control or the distribution of the non-deterministic external force change in time intervals relevant for wear, then $G$ may depend on the time $t$, too.

If the exogenous conditions are unchanged, then the wear evolution is modeled by the autonomous evolution equation

$$\dot{v} = G(v), \quad v(0) = u_0 \quad (15)$$

in the slow time scale. Eq. (15) is the result of the decoupling of the time scales. The fast time scale, i.e. the dynamical behavior of the system is contained in $G(v)$ by Eqs. (13) and (14).

Every evaluation of the right-hand side of Eq. (15) contains the computation of the wear density via a simulation of the dynamical behavior for frozen geometry. Since the influence of the short-time dynamics on the wear speed is small, Eq. (15) models the wear evolution in its characteristic time scale, and $v(t, \sigma) \approx u(t, \sigma)$ is a close approximation.

On the one hand, a numerical solution of Eq. (15) can be seen as an approximation on the macro-grid within the heterogeneous multiscale method from [8], and the presented method for decoupling the time scales is a specification of this general framework. On the other hand, Eq. (15) formulates a well-posed problem by considering the wear density instead of regarding the punctual wear in system (7, 8). Hence, it continues the considerations in [10,11].

In practical calculations, the same approach as in Eq. (13) serves for the determination of the mean wear speed, namely

$$G(u)(\sigma) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T g(y(t), u)(\sigma) \, dt, \quad (16)$$
which is approximated by
\[ G[t_{med}] \approx \frac{1}{t_{med}} \int_0^{t_{med}} g(y(t), u(\sigma)) \, dt = G[t_{med}] \approx G(u) \]
with a suitable medium time \( t_{med} \), cf. Section 3.3.

3.3. Numerical remarks

The most obvious advantage of the evolution equation (15) lies in the decoupling of the wear evolution from the dynamical behavior of the wearing system. Eq. (15) can be scaled, i.e. with \( \tau = t/\Delta t \) and \( v'(\tau, \sigma) = v(t, \sigma) \), the evolution equation reads
\[ \frac{\partial}{\partial \tau} v'(\tau, \cdot) = aG(v'(\tau, \cdot)), \quad v'(0) = u_0. \]  
(18)
Now, the scale \( \tau \) can be adapted to the time scale of typical wear processes for the numerical integration of Eq. (15).

A more crucial question is the determination of \( t_{med} \) in Eq. (17). We can use an extrapolation scheme. Since Eq. (16) matches the estimation of the expectation by the mean value of a sample, we know
\[ |G[t_{med}](u(\sigma)) - G(u(\sigma))| = O\left(\frac{1}{\sqrt{t_{med}}}\right) \text{ for } t_{med} \to \infty \]
and a constant \( C \) exists, which fulfills \( G[t_{med}](u(\sigma)) - G(u(\sigma)) \approx C/\sqrt{t_{med}} \). That yields
\[ G[t_1](u(\sigma)) - G[t_2](u(\sigma)) \approx C\left(\frac{1}{\sqrt{t_1}} - \frac{1}{\sqrt{t_2}}\right) \]
and thus an approximation of \( C \). Now, the time \( t_{med} \) fulfilling \( C/\sqrt{t_{med}} < \varepsilon \) for all \( \sigma \in S \) with a given error bound \( \varepsilon \) can easily be estimated. That leads to an adaptive method for the determination of the time interval \( t_{med} \).

In practical calculations, \( G \) is computed by Eq. (17), and the measures \( v_Y \) and \( v_U \) need not to be determined.

In this section, we have dealt with a one-dimensional description of the wear by the accumulated material loss \( u \in \mathbb{R} \). Our approach can be applied for a vector-valued wear description, which additionally contains e.g. material degradation, fatigue strength or micro-crack density. In this case, the function \( g \) in Eq. (5) maps into a space \( U \) of vector-valued functions, and so does \( G \) in Eq. (15). The presented method of decoupled time scales works in full analogy.

4. Disk wheel model

4.1. Wearing disk wheel

We consider a two-dimensional disk with an angle-dependent radius. It rolls over a flat rail, driven by a moment \( M \) and exposed to a drag force.

The disk wheel has the one-dimensional wearing surface
\[ \mathcal{U}(t) = \{(r(\sigma) + u(t, \sigma))(\sin \sigma, -\cos \sigma)^T, \sigma \in [0, 2\pi]\} \subset \mathbb{R}^2, \]
which is its circumference. Hence, we have the parameters \( S = [0, 2\pi] \subset \mathbb{R}^1 \), and the reference surface is \( \mathcal{U}_0 = \{r(\sigma)(\sin \sigma, -\cos \sigma)^T, \sigma \in [0, 2\pi]\} \). The space of all admissible wear states are all twice differentiable \( 2\pi \)-periodic functions \( U = C^2_{2\pi} \).

We describe the dynamics of our disk wheel by three degrees of freedom. These are the displacement \( x \) in traveling direction, the vertical displacement \( z \) and the rotational angle \( \varphi \) of the disk. The dynamical state is \( y = (q, \dot{q})^T \) with \( q = (x, z, \varphi) \in \mathbb{R}^3, \dot{b} = 3 \). The disk is exposed to a driving moment \( M = M(t) \), a normal load \( F_G \), which models the load of a railway carriage, and a drag force \( F_C \), e.g. an aerodynamic resistance.

Without loss of generality, the angle \( \varphi = 0 \) describes the position of the wheel, where \( \sigma = 0 \) denotes a particle at the wheel circumference vertically below the hub. Then, an angle \( \varphi \) means that the particle \( \vartheta = -\varphi \) is directly below the hub, cf. Fig. 2. We assume that contact between wheel and rail occurs only at this point. The rail-wheel contact
interaction is modeled by a damped Hertzian spring. It causes a normal force $F_N$ and together with a creepage a frictional tangential force $F_T$ at $\vartheta = -\varphi$.

With the mass $m$, the moment of inertia $J$ and the gravitational acceleration $\gamma = -9.81 \text{ m s}^{-2}$, the equations of motion read in components

$$m \ddot{x} = F_T + F_C, \quad (19)$$
$$m \ddot{z} = F_G + m\gamma + F_N, \quad (20)$$
$$J \ddot{\varphi} = M + [r(-\varphi) + u(t, -\varphi)]F_T. \quad (21)$$

Let us remark that Eq. (21) contains the radius $r(-\varphi) + u(t, -\varphi)$ vertically below the hub. As long as $u$, and therefore the material loss, is very small compared to the radius $r$, this is an acceptable approximation of the lever arm in the braking moment caused by the friction force.

With the same approximation we calculate the slip $s$, i.e. the relative velocity of particles in contact, and we get

$$s = \dot{x} + [r(-\varphi) + u(t, -\varphi)]\dot{\varphi}. \quad (22)$$

Rolling friction [18] implies the friction force

$$F_T = -\mu F_N s. \quad (23)$$

Except for extreme initial conditions or control motion, the friction force acts into the same directions as the traveling speed. In fact, it is the driving force.

Furthermore, the damped Hertzian spring [13] simulates an elastic contact between disk wheel and rail, and we get

$$F_N = \left[k_{\text{Hertz}}(r(-\varphi) + u(t, -\varphi) - z)^{3/2} - d\ddot{z}\right]^+ \quad (24)$$

with the Hertzian spring constant $k_{\text{Hertz}}$, the damping constant $d$ and the positive part $[\cdot]^+$ because the normal force vanishes in the case that the wheel lifts off. Finally, let the drag force be proportional to the square of the velocity and act opposite to the direction of motion. With the drag coefficient $c$, it is

$$F_C = -c|\dot{x}||x|. \quad (25)$$

Until now, the disk wheel model is a rigid-body model. That leads to a single point contact, which is not suitable for a wear simulation, cf. Remark 1. Therefore, we follow Eq. (4) and use a footprint $\chi$ to distribute the concentrated forces $F_N$ and $F_T$ over an assumed contact area. We get a disk wheel model with a continuous dissipated frictional power density. The normal force density is $f_N(t, \sigma) = \chi(\sigma + \varphi(t))F_N$. 

---

**Fig. 2.** Disk wheel model: the moment $M$ drives the wheel against a drag force $F_C$ and a friction force $F_T$. The friction force acts in the contact area and yields a moment depending on the leverage, i.e. the radius $r(\vartheta) + u(t, \vartheta)$ with $\vartheta = -\varphi$. The vertical load $F_G$ models the weight of the carriage. The vertical motion of the wheel is governed by a damped Hertzian spring replacing the contact mechanics.
With the slip \( s \in \mathbb{R} \) from Eq. (22), we find the frictional force density \( f_\tau(t, \sigma) = \chi(\sigma + \varphi(t))F_\tau \) by Eq. (23). Archard’s law (6) now reads
\[
\dot{u}(t, \sigma) = \beta s f_\tau(t, \sigma) = \chi(\sigma + \varphi(t)) \cdot \beta s F_\tau.
\]

### 4.2. Stationary rolling

Stationary rolling is a quasi-stationary state with vanishing accelerations in Eqs. (19)–(21). It requires a constant radius of the wheel. We set \( r(\sigma) = r \) and \( u = 0 \). Let be \( M < 0 \), which makes the wheel turn in mathematically negative sense and the traveling speed of the wheel positive.

We denote the velocities, forces etc. in the stationary rolling by a star. Eqs. (19) and (21) yield
\[
F_\tau^* = -\frac{M}{r} \quad \text{and} \quad (\dot{x}^*)^2 = \frac{F_\tau^*}{c}.
\]

The stationary normal force \( F_N^* \) balances the load and the weight of the wheel in Eq. (20), and \( \dot{z} = 0 \) in Eq. (24) yields
\[
F_N^* = -F_G - my\gamma. \quad \text{Finally, Eq. (23) provides}
\]
\[
s^* = -\frac{F_\tau^*}{\mu F_N^*}. \quad (25)
\]

The total dissipated power \( W^* \) in the contact is proportional to the wear speed by Archard’s law (6) and reads
\[
W^* = s^* F_\tau^* = -\frac{(F_\tau^*)^2}{\mu F_N^*} = -\frac{M^2}{r^2 \mu F_N^*} = -\frac{\mu F_N^*}{\mu F_N^*}. \quad (26)
\]

Following Eq. (26), the total dissipated power in the contact is proportional to the square of the driving moment or to the fourth power of the traveling speed \( \dot{x} \) of the wheel. Since an initially perfectly round wheel stays perfectly round, we get
\[
-u(t, \sigma) \sim t \dot{x}^4. \quad (27)
\]

Eq. (27) can be used to estimate the wear coefficient \( \beta \) from given data.

### 5. Results

We will observe wear evolutions with nearly flat geometries and compare them with wear processes leading to shape oscillations in \( \sigma \in S \). These oscillations over the circumference are referred to as polygonalization of the wheel [5,18,19].

#### 5.1. Example set-up

In the numerical examples, the geometry \( u(t, \sigma) \) is described by a periodic B-spline approximation
\[
u(t, \sigma) = \sum_{i=1}^{N} \alpha_{i}(t)B_4(\sigma)
\]
of fourth order with \( N = 120 \) grid points in angular direction. Thus the geometry is is twice continuously differentiable. The spline coefficients \( \alpha_{i}(t), \ell = 1, \ldots, N \) contain the condensed information about the wear evolution.

A non-deterministic exogenous influence is modeled here by an initial geometry of the disk, which is not perfectly round. We introduce a small indentation of 2 \( \mu \)m depth by setting the four first spline coefficients to \( \alpha_{1}(0) = \alpha_{4}(0) = -10^{-6} \), \( \alpha_{2}(0) = \alpha_{3}(0) = -2 \times 10^{-6} \) in the initial geometry \( u(0, \sigma) \). All other splines coefficients are initially set to zero. The indentation is found near \( \sigma = 0.1 \), cf. the following figures. By the way, such an indentation is in the range of the measurement accuracy of extended bodies and would never be discovered in a realistic test of a railway wheel.

**Example 1.** In the model set-up, the mass of the wheel is \( m = 300 \) kg, its radius is \( r = 0.5 \) m, and its momentum of inertia is \( J = 37.5 \) \( \text{kg m}^2 \). The load \( F_G = -50 \times 10^3 \) N represents a wagon with about 5 t per wheel. The driving moment is chosen between \( M = -200, \ldots, -300 \) Nm so that mean driving velocities \( v^* = 30, \ldots, 35 \) m/s occur.
The stiffness of the Hertzian spring is calculated by the slightly exaggerated assumption that under the given static load the wheel center approaches the support by 1 mm. That results in \( k_{\text{Hertz}} = 1.6742 \times 10^9 \text{ Nm}^{2/3} \). The damping of the spring is rather small and set to \( d = 100 \text{ Ns/m} \). We use \( \mu = 0.2 \text{ s/m} \), \( c = 0.5 \text{ Ns}/\text{m}^2 \) and \( \beta = 0.75 \times 10^{-12} \text{ m/N} \).

The data of Example 1 are in accordance to present high-speed trains. The more realistic speed of \( v = 55.5 \text{ m/s} \) requires \( M = -770 \text{ Nm} \). Then, Eq. (25) leads to \( s^* = -0.145 \text{ m/s} \) or a relative slip of 0.3 \%, the rotational speed is \( \psi^* = -111.3 \text{ s}^{-1} \), and the wheel turns 17.7 times per second. The total dissipated power in the contact is \( W^* = s^* F_T^* = -224 \text{ Nm/s} \). Hence, we get a material loss \( -u = 3 \times 10^{-12} \text{ m} \) during one contact event, i.e. per rotation.

On the other hand, after about 150,000 km of traveling, the wheel has lost a material layer of about 150 \( \mu \text{m} \) thickness, see Section 1. A simple division gives a material loss of \( 3 \times 10^{-12} \text{ m} \) per rotation, which is much less than the atomic radius of the material. Of course, this last observation motivates some critics on a continuous modeling of wear processes.

5.2. Wear simulation with decoupled time scales

Figs. 3 and 4 show numerical simulation results of wear evolutions for the different driving moments \( M \in \{-230, -240, -250, -260\} \text{ Nm} \) and thus for different traveling speed in the small range 30.2, . . . , 32.2 \text{ m/s}, what is rather slow compared to high-speed trains. The total number of wheel revolutions is 800 million, which is shown in Figs. 3 and 4 by 20 steps of 40 million each.

It is clearly visible that the mean wear increases with growing driving speed. Furthermore, the initial indentation, which is marked by a gray circle in the plots, results in qualitatively different shapes. Whereas Fig. 3, left, shows smooth wear formations with a small oscillation in the radius, Fig. 3, right, contains a much more bumpy wear formation. The bumpy wear formation occurs in Fig. 4, left, in a fast growing manner and after a shorter time period of rolling, polygonalization occurs. Surprisingly, after the development of the bumpy wear formation, its shape is conserved over
Fig. 5. Left: dependence of the mean wear $u_{\text{mean}}$ on the driving moment $M$ after $t = 0, 10, 20, \ldots, 100 \times 10^6$ s (alternating solid and dashed lines). Right: dependence of the polygonalization measure $D$ on the driving moment $M$ for $t = 0, 20, \ldots, 100 \times 10^6$ s. The non-monotonous behavior is clearly visible. Certain velocities (vertical dashed lines) cause resonance effects between vertical and rotational motion.

a long time period. A higher driving speed in 4, right, leads again to a smoother but still wavy wear formation. Section 5.3 is devoted to a more systematic study of the relation between traveling speed and the wear formation.

Additionally, in Fig. 4, we remark that the wear shape moves in angular direction over the circumference of the wheel. The wheel rolls in mathematically negative sense, and thus, the wheel-fixed contact angle $\sigma$ increases. Slightly behind a maximum of the radius, the dissipated power density reaches a maximum, because the disk wheel falls onto the support and the normal force is increased. This phase shift effect may lead to a self-amplification of wear irregularities.

5.3. Speed dependence

We introduce two integral measures

$$u_{\text{mean}}(t) = \frac{1}{2\pi} \int_0^{2\pi} u(t, \varphi) \, d\varphi, \quad D(t) = \int_0^{2\pi} u_{\varphi}(t, \varphi)^2 \, d\varphi,$$

namely the average radius reduction $u_{\text{mean}}(t)$ and $D(t)$ as a measure of the oscillation in the wear formation. The measure $D$ expresses, how wavy or bumpy the wear formation is, i.e. the strength of polygonalization.

Fig. 5, left, shows the mean wear $u_{\text{mean}}$. We remark that the mean wear depends monotonously on the driving moment and thus on the speed and that the mean wear increases nearly linearly with time, compare Eq. (27).

In contrast to that, the oscillation measure $D$ shows a completely non-monotonous behavior in Fig. 5, right. There are ranges of the driving moment, which generate wavy wear formations, and they are separated by ranges with very smooth wear formations, i.e., where $D$ nearly vanishes.

Resonance effects between the vertical motion and the rotation occur if the eigenfrequency $\omega_{\text{ver}}$ of the vertical motion is an integer multiple of the rotational frequency $\omega_{\text{rot}}$. We linearize the Hertzian spring for $u = 0$ by

$$k_{\text{Hertz}}(r - z)^{3/2} = k_{\text{Hertz}}(r - z^*)^{3/2} - k_{\text{lin}}(z - z^*) + O(z - z^*)^2$$

with $k_{\text{lin}} = \frac{3}{2}k_{\text{Hertz}}\sqrt{r - z}$. Then, Eq. (20) yields

$$\omega_{\text{ver}} \approx \frac{1}{2\pi} \sqrt{\frac{k_{\text{lin}}}{m}}$$

as long as the squared damping constant $d^2$ is small compared to $k_{\text{lin}}$, which is fulfilled by the values of Example 1. Eq. (28) is an approximation only because the oscillation of the nonlinear Hertzian spring is not harmonic and damping is not taken into consideration. The frequency of the rotational motion is

$$\omega_{\text{rot}} = \phi^* = \frac{\dot{x}^*}{2\pi r} \approx \frac{\ddot{x}^*}{2\pi r} = \frac{1}{2\pi} \sqrt{\frac{M}{r^3 c}}.$$
Now, the relation $\omega_{\text{ver}} = N \omega_{\text{rot}}$ with $N \in \mathbb{N}$ yields

$$-M = \frac{1}{N^2} \cdot \frac{r^3 c \kappa_{\text{lin}}}{m}.$$  \hfill (29)

The dashed vertical lines in Fig. 5, right, indicate the driving moments with resonance properties for $N = 9$ and $N = 8$ in Eq. (29), respectively.

Obviously, resonance is not the only cause of the growth of wavy patterns. It is affected by the permanent phase transition of the wavy wear formations, too.

Fig. 6 extracts the wear evolution for the driving moment, where the oscillation measure $D$ is maximal. Eight humps or nine humps in the polygonalized wear formation are visible, cf. $N = 8$ or $N = 9$ in Eq. (29).

Let us remark that $M \approx -1800 \text{ N}$ leading to $v \approx 85 \text{ m/s}$, evokes a wear formation with three humps – as observed in high-speed trains [14,20,21], and that this value of $v$ is in the range of the maximal speed of present high-speed trains allowed in regular service.

5.4. Resmoothing

The preceding subsections deal with the wear evolution in the case of constant driving moments. Of course, this is not a likely scenario for any real railway operation. Here, we give a simple example of the wear evolution for a variable traveling speed.

This example starts with a wavy wear formation induced by the constant driving moment $M = -250 \text{ Nm}$, cf. Fig. 4, left. After this starting phase with half the simulation time used in the previous examples, the driving moment is changed to a value, which does not lead to polygonalized shapes if it is constantly applied. In the first case, the driving moment is changed to $M = -220 \text{ Nm}$, i.e. the speed is decreased, cf. Fig. 7, left. In the second case, the driving moment is then changed to $M = -270 \text{ Nm}$ corresponding to an increased speed, cf. Fig. 7, right.

Fig. 7. Initial wear formations can be amplified (left) or resmoothed (right) by suitable driving moments.
Fig. 8. Left: typical dissipated power $W \sim -sF_T$ (positive sign) in the contact $\sigma = -\varphi$ for a not perfectly round wheel with initial indentation. Four cycles of the wheel are shown. Right: first revolutions of the wearing wheel with relative initial slip $s/v = 0.005$, $M = -225$ and $\varphi(0) = \pi$. Two perturbations of similar size are visible, the one at $\sigma = \pi$ is caused by the relatively large initial slip, which is quickly damped out. Four million rotations are shown, 400,000 between two neighbored lines, which reduce to 2 due to the amplification factor $a = 2 \times 10^6$.

In the first case, we observe a further amplification of the waviness after the change, although the same driving moment $M = -220$ Nm, constantly applied, would not lead to wavy wear formations, cf. Fig. 5, right. In the second case, the wavy wear formation can be resmoothed by the moment $M = -270$ Nm. Of course, the mean wear is higher for higher speed.

This example shows that the mean driving moment or the traveling speed, respectively, are not the only influence factors determining the qualitative behavior of the wear formation. Variable speed may lead to unexpected wear evolutions, and the process is still not completely understood.

5.5. Robustness of the simulation results

The wear formations described in the previous subsections do not depend on the particular shape of the small initial indentation or a more general small initial out-of-roundness. Fig. 4 shows the typical behaviour that first the initial formation spreads out over the circumference and gets some periodicity and then suddenly grows up and finds a typical wear formation, which is conserved for a longer interval of the wear evolution. Various numerical tests confirm this observation.

Furthermore, the simulation results are conserved if the model is extended by track irregularities, which can be seen as non-deterministic disturbances in the slow time scale of the wear evolution. Hence, they behave like the non-deterministic external force $F_{\text{ext}}(t)$ in Eq. (2). In analogy to the introduction of the stiff contact spring, indeed track irregularities or other external influences can be modeled by additional non-deterministic forces in the contact, which enter the determination of the measure $\nu_Y$ in Eq. (13).

6. Comparison with the handling with amplified wear coefficient

The dynamical behavior of the rather simple mechanical system of a rigid disk wheel (19)–(21) can be simulated by standard methods for stiff ordinary differential equation with high accuracy even for long time intervals [7]. Thus, we can compare different simulation techniques, in particular here simulations with amplified wear coefficient for different $a$, what would not be possible for a more sophisticated wheel model like in [19].

Fig. 8, left, shows the dissipated friction power in the contact point of a disk wheel with the 2 $\mu$m-indentation. It is clearly visible that the dissipated friction power is not periodic for the chosen initial state. Periodic behavior can be regarded as exception.

Fig. 8, right, shows the beginning of the traditional wear simulation with amplified wear coefficient. The simulation is started with a wheel at the rotational angle $\varphi(0) = \pi$ and a non-vanishing and slightly enlarged slip. This initial slip causes a large friction power at the contact point at $\sigma = -\varphi = \pi \mod 2\pi$ and thus an additional indentation, which would be very tiny for a realistic wear coefficient $\beta$. But due to the amplified wear coefficient $\beta a$, the additional indentation reaches a depth of about 2 $\mu$m, too. After the generation of the additional indentation, the slip normalizes and the radius diminishes nearly constantly over the angle and the two indentations.
Fig. 9. Left: numerical simulation of the wearing radius with the traditional method and amplification factor $a = 2 \times 10^5$ (4000 rotations in the total numerical calculation). The additional indentation at $\sigma = \pi$ is still visible and evolves to an erroneous slight polygonalization. Right: same with amplification factor $a = 2 \times 10^6$ (400 rotations). Very erroneous and artificial oscillations occur. The wear behavior is qualitatively changed.

If the simulation is continued from the last wear state of Fig. 8, right, with decoupled time scales like in Section 5.2, cf. Fig. 3, or with much lower amplification factor $a$, then the radius further diminishes nearly constantly over the angle and the two indentations.

Fig. 9 gives a completely different impression. Here, the simulation is continued with rather large amplification factors $a = 2 \times 10^5$ (left) and $a = 2 \times 10^6$ (right), so that only 4000 and 400 revolutions, respectively, needed to be computed. Indeed, the range of the total material loss is similar. But both plots show a further amplification of the additional indentation. Whereas the left plot of Fig. 9 shows a polygonalization of the wheel, the right plot gives a steep jump in the radius near $\sigma = \pi$, which is smoothed a little during the later period of the simulation. Let us mention that the false polygonalization in the left plot appears to be more dangerous because it might be taken for a realistic simulation result, but it is not, as shown in Section 5.2, Fig. 3.

7. Conclusion

We have presented a mathematical formulation for an evolution problem comprising a dynamical model and a wear model. For applications, e.g. in railway mechanics, longtime integration of the geometrical changes is needed but not practically available because the system of evolution equations couples two completely different time scales.

The introduction of a wear density decouples the time scales, and the numerical integration of the wear evolution becomes a straightforward standard numerical problem. We demonstrated our approach on the example of a disk wheel model. Parameters were chosen to be consistent with phenomena observed at the wheels of high-speed trains.

For the studied model, wear evolution was apparently dominated by resonance between vertical and rotational motion. The traveling speed, which is determined by the driving moment in the stationary regime, decides about the character of the shape evolution. For different speeds, we may obtain patterns with different wave numbers, which may be evolving fast or be suppressed again by resmoothing effects, if speed is changed. Patterns may travel around the circumference while evolving. As a rule one can say that speed control maintaining a constant traveling speed is highly risky for almost all values in the high speed range.

A more thorough mathematical analysis of the wear density and thus the type of the evolution problem is needed and planned for a future paper. In particular, the sensitivity to eigenfrequencies of the contact model and the possibility of controlled resmoothing hold scientific as well as practical potential for further research.

References