Abstract

This paper describes a biomechanical model for numerical simulation of front and back somersaults, without twist, performed on the trampoline. The developed mathematical formulation is used to solve an inverse dynamics problem, in which the moments of muscle forces at the joints that result in a given (measured) motion are determined. The nature of the stunts and the way the human body is maneuvered and controlled can be studied. The calculated torques can then be used as control signals for a dynamic simulation. This provides a way to check the inverse dynamics procedures, and influence of typical control errors on somersault performance can be studied. To achieve these goals, the nonlinear dynamical model of the trampolinist and the interacting trampoline bed has been identified, and a methodology for recording the actual somersault performances was proposed. Some results of numerical simulations are reported.

Keywords: Trampoline somersaults; Human movement modeling and identification; Inverse dynamics

1. Introduction

Front and back somersaults are common acrobatic stunts performed on the trampoline. However, even the easiest (tucked) somersaults are usually recognized only qualitatively as concerns some general guiding/teaching rules for their correct performance. The nature of the stunts and the way the human body is maneuvered and controlled has not so far been well understood. The quantitative description may be useful for both cognitive and practical reasons, leading to better understanding of the athlete movements and making a basis for more conscious mastering of the somersault evolutions. The present contribution is an attempt towards an extensive study on the trampoline-performed human maneuvers, beginning with the kinematic analysis based on the recorded (filmed) actual somersault performances, and getting through the inverse dynamics problem solution and the direct dynamics simulation. The latter two tasks must involve a nonlinear dynamical model of the trampolinist and the (recurrently) interacting trampoline bed.

The inverse dynamics problem (Chao and Rim, 1973; García de Jalón and Bayo, 1993) aims at determining the applied torques (moments of muscle forces) at the joints that result in a given (observed) motion of a biomechanical system. By comparing different inverse dynamics solutions, obtained for the motion characteristics of somersaults ranging from correct to incorrect, more strict rules of performance and control of the stunts can be understood. The inverse dynamics solutions can also be used as initial guesses for optimization algorithms—searching among control signals that result in coordinated movement (Chao and Rim, 1973; Tashman et al., 1995). The dynamic simulation (direct dynamics problem) in which the calculated control is used as the input signal is then referred to as inverse dynamics simulation. For the same biomechanical model used, the outputs of the inverse dynamics simulation should approximately match the movement pattern that served as input to the inverse dynamics problem. In this way, the direct dynamics problem provides a way to check the inverse dynamics procedures and vice versa. Then, assuming that the use of the inverse dynamics solution in the dynamic simulation faithfully replicates the data used to produce the inverse dynamics solution, the influence of typical control errors on somersault performance can be studied. This can be done by comparing...
outputs to intentionally modified control inputs with those obtained from the inverse dynamics solution.

The inverse dynamics simulation does not often result in calculated outputs that exactly match the observed data, and, when the disagreement is remarkable, the phenomenon is termed inverse dynamics simulation failure. Tashman et al. (1995) and Zajac (1993) suggested that the possible causes might be the measurement and derivative estimation errors, unfit inverse dynamics and direct dynamics models used, and also the numerical integration errors. Risher et al. (1997) and Kuo (1998) argued however that these grounds for the inverse dynamics simulation failure can be minimized. Then Risher et al. (1997) pointed at a new contributor to the failure, termed insufficient control signal dimensionality. They demonstrated that the replication of the inverse dynamics simulation results, compared to the data used as input to inverse dynamics problem, is highly dependent on the way the discrete controls determined via inverse dynamics are mapped into the continuous time functions of controls used in the direct dynamics. It was then stated that the mappings with insufficient dimensionality (such as linear interpolation between discrete controls) never assure the replication, and a method for choosing additional parameters to specify the inverse dynamics simulation failure-free mappings was proposed. In our study we used cubic spline functions for the mapping, and we achieved a good consistency between the inverse dynamics simulation and the data used as input to the inverse dynamics problem.

Reliability of solutions to both the inverse dynamics problem and the direct dynamics simulation requires an adequate model of the trampolinist and interacting trampoline bed. The model characteristics such as the bed stiffness, human body mass, geometrical parameters, etc. need to be identified. Then, the observed somersault performances must be recorded (filmed), and the obtained kinematic characteristics must be numerically recalculated to a required form of input data for the inverse dynamics problem. A peculiarity of the developed mathematical model is that we deal with a 9-degrees-of-freedom rigid multibody system with six control signals. The dimensional incompatibility must then be reflected in the proposed algorithms for the numerical solutions of the inverse dynamics problem and the direct dynamics simulation. All these issues are addressed in the sequel of this contribution.

2. The multibody model

The trampoline somersaults are difficult to model for many reasons. They generally include a space nature of motion, a very sophisticated description of human body, its control and movement, and a complex and recurrent interaction from the bed. The following limitations have thus been assumed in the present study:

- Planar motion is considered. This limits our analysis to the front and back somersaults (in tuck, pike and straight positions) without twisting.
- The trampolinist is modeled as a 9-degrees-of-freedom rigid multibody system shown in Fig. 1. Synchronous motions of both two legs and two arms are assumed.
- The trampoline bed is modeled as weightless canvas of known (measured) stiffness and damping characteristics.

Let us denote that a somersault begins/ends the instant the athlete drops on the trampoline and touches the bed with his feet. The jump can then be divided into two main phases: the support phase, when the feet push on the bed, and the flying phase, when the trampolinist does not touch the bed. In the support phase, the downward motion of the body is first decelerated, and then the athlete rebounds upward. During this period, a correct somersault must be initiated by appropriately segmenting the upper and lower body. By segmentation we mean that for front somersaults the upper body bends forward and downward (relative to the center of gravity) and the hips move upward and backward. This allows the force generated by the feet pushing on the trampoline to effectively push the body parts around the center of gravity resulting in a front somersault. A back somersault is done the same way except the upper body
where the vertical deflection reactions from the trampoline are dependent primarily for a sample trampoline, and draw a conclusion that theistics. We measured (see Appendix A) the characteristics with stiffness and damping characteristics. We measured (see Appendix A) the characteristics for a sample trampoline, and draw a conclusion that the reactions from the trampoline are dependent primarily on the vertical deflection \( y_A \) (see Fig. 2). The more precise (nonlinear) dependencies are

\[
R_x = -k_x x_A - c_x \dot{x}_A,
\]

\[
R_y = -(a y_A^2 + b y_A) - (c y_A^2 + d y_A) \dot{y}_A,
\]

\[
M_A = -(c y_A + f) \theta_4 - c_0 \dot{\theta}_4,
\]

where \( \theta_4 = \pi/2 - \varphi_A \). The coefficients \( k_x, c_x, c_0 \) do not approximately depend on \( y_A \), and the constants \( a-f \) that define the nonlinear dependences on \( y_A \) are reported in Eqs. (A.1) and (A.2) of Appendix A.

The position of the 9-degrees-of-freedom trampoline multibody model can be explicitly described by nine generalized coordinates \( \mathbf{q} = [x_H \ y_H \ \varphi_1 \ \varphi_2 \ \varphi_3 \ \varphi_4 \ \varphi_5 \ \varphi_6 \ \varphi_7]^T \), where \( x_H \) and \( y_H \) are the hip coordinates in the inertial reference frame \( xy \), and the angular coordinates \( \varphi_i \ (i = 1, ..., 7) \) shown in Fig. 1 are all measured from the vertical direction; \( \mathbf{q} \) are thus the absolute coordinates. The multibody system is then controlled by six torques \( \mathbf{\tau} = [\tau_1 \ \tau_2 \ \tau_3 \ \tau_4 \ \tau_5 \ \tau_6]^T \) that model the moments of muscle forces at the joints. As it will be discussed later on, the difference in the freedom and control input dimensions, \( 9 - 6 = 3 \), is reflected in the proposed algorithms for the numerical solutions of the inverse dynamics problem and the direct dynamics simulation.

Following the multibody codes described by García de Jalón and Bayo (1993), Blajer (1997), and Schiehlen (1997), the dynamic equations for the modeled trampoline can be obtained in the following generic form (see also, e.g. Blajer and Schiehlen, 1992; Risher et al., 1997, Kuo, 1998; Spägel et al., 1999)

\[
\mathbf{M(q)} \ddot{\mathbf{q}} + \mathbf{d(q, \dot{q})} = \mathbf{f(q)} + \mathbf{r(q)} + \mathbf{B}^T \mathbf{\tau},
\]

where \( \mathbf{M} \) is the \( 9 \times 9 \) generalized mass matrix, the \( 9 \)-vector \( \mathbf{d} \) contains the dynamic generalized forces due to the centrifugal accelerations, \( \mathbf{f} \) and \( \mathbf{r} \) are similar generalized force vectors resulting from the gravitational forces and reactions from the trampoline bed (\( r = \theta \) in the flying phase), and \( \mathbf{B}^T \) is the \( 9 \times 6 \) (constant) matrix of control input distribution. The explicit form of the equations is reported in Appendix B.

3. Recording of actual somersault performances

The solution of inverse dynamics problem needs to introduce the motion characteristics \( \mathbf{q}_d(t) \), \( \dot{\mathbf{q}}_d(t) \) and \( \ddot{\mathbf{q}}_d(t) \). Within this work these are measured from the actual somersault performances using a camera with 120 Hz shutter frequency. The jumper in the photographs of Fig. 3 is a physical education student who has undergone one-semester training on the trampoline. The positions of markers placed on the athlete’s leg, hip, trunk, head and arms are tracked, and from these measurements discrete trajectories \( \mathbf{q}_d(t) \) are obtained by using a specialized computer program. The cubic spline interpolation according to Reinsch (1967) is then used to
generate continuous $q_d(t)$ consistent with $q^*_d(t)$. The analytically given functions $q_d(t)$ are finally differentiated to obtain $\dot{q}_d(t)$ and $\ddot{q}_d(t)$. Since the end points of the second derivative of the cubic spline are constrained to zero, which may produce an error in the solution of inverse simulation problem, we used few points more of measured $q_d$, coming before and after the simulation time period.

In the flying phase the precision of determination of $q_d(t)$ can be verified using the axiom that the trampoline gravity center must move along a parabola (or vertically). Stated another way, the acceleration of the gravity center in the $x$ and $y$ directions must be equal to zero and to the acceleration of gravity, respectively. Then, the total angular momentum of the body with respect to the gravity center must remain constant. These conditions are not valid during the support phase, however. To achieve high precision of determination of $x^*_{Ad}(t)$, $y^*_{Ad}(t)$ and $\theta^*_{ad}(t)$ during this phase, required to precisely estimate the reactions from the bed according to Eqs. (1), (A.1) and (A.2), we used a shadow registration method and the assisted high-precision equipment developed by Dziewiecki and Karpiłowski (1999). The same cubic spline interpolation is then used to generate continuous $x_{Ad}(t)$, $y_{Ad}(t)$ and $\theta_{ad}(t)$ consistent with $x^*_{Ad}(t)$, $y^*_{Ad}(t)$ and $\theta^*_{ad}(t)$, and these analytically given functions are then differentiated to obtain $\dot{x}_{Ad}(t)$, $\dot{y}_{Ad}(t)$ and $\dot{\theta}_{ad}(t)$ as used in Eq. (1).
In Fig. 4 some selected characteristics of the trampolinist motion during the measured somersault are shown. The characteristics provide a basis for studying the nature of the stunt—the way the human body is maneuvered. The reported data demonstrate the variations in time of the hip point $H$ position, $x_H^*(t)$ and $y_H^*(t)$, and the relative angles that specify the angular orientation of the body segments to each other in the joints $H$, $K$, and $S$, respectively: $\phi_2^*(t) - \phi_1^*(t)$, $\phi_3^*(t) - \phi_2^*(t)$ and $\phi_5^*(t) - \phi_1^*(t)$. The trajectory of the gravity center during the flying phase, $y_C^*(x_C^*)$, is also shown to approve the precision of the measured data. The condition of conservation of the total angular momentum of the system is achieved as well. The sample motion characteristics of $q_d(t)$, $\dot{q}_d(t)$ and $\ddot{q}_d(t)$, used for the inverse dynamics solution, are then reported in Fig. 5.

Fig. 4. Selected motion characteristics of the measured somersault in pike position.

Fig. 5. Selected motion characteristics used for the inverse dynamics solution.
The accuracy of the characteristics at the acceleration level is of special importance for the solution validity.

Using the recorded/estimated characteristics \( x_{Ad}(t), y_{Ad}(t), \dot{x}_{Ad}(t), \) and \( \ddot{x}_{Ad}(t) \), the reactions from the trampoline bed during the support phase were determined from Eq. (1), where the appropriate constants are reported in Eqs. (A.1) and (A.2) of Appendix A. The estimated trampoline reactions are shown in Fig. 6.

4. The inverse dynamics problem

Eq. (2) stands for nine dynamic motion equations dependent on six controls \( t \). In other words, in the 9-dimensional configuration space of the system, only six directions are controlled while the other three remain uncontrolled. The configuration space can then be split into controlled and uncontrolled subspaces of dimensions 6 and 3, respectively. While the controlled subspace is defined by vectors represented as rows (in columns of \( B \) defined in Eq. (B.2d)), the uncontrolled subspace can be spanned by vectors represented as columns of a \( 9 \times 3 \)-dimensional matrix \( D \)—an orthogonal complement matrix to \( B \), i.e. \( D^T B = 0 \) (see Blajer, 1997). For the case at hand we propose

\[
D = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}.
\] (3)

The premultiplication of Eq. (2) by the \( 9 \times 9 \) matrix \( [M^{-1}B^T : D]^T \) yields the projections of the dynamic equations into the controlled and uncontrolled subspaces, respectively,

\[
Bq + BM^{-1}d = BM^{-1}f + BM^{-1}r
+ BM^{-1}B^T t,
\] (4)

\[
D^T M \ddot{q} + D^T d = D^T f + D^T r.
\] (5)

By applying the specified motion characteristics \( \dot{q}_d(t) \), \( \ddot{q}_d(t) \), and \( \dddot{q}_d(t) \), the control \( t_d(t) \) that assure the realization of the specified motion can be determined from Eq. (4) manipulated to

\[
\tau_d(t) = (BM^{-1}B^T)^{-1}B \times [\dot{q}_d + \dddot{q}_d^{-1}(d - \ddot{r} - \ddot{f})],
\] (6)

where \( \dddot{q}_d = M(q_d) \), \( \dddot{d} = d(q_d, q_d) \), \( \dddot{r} = r(q_d, x_{Ad}, y_{Ad}, \theta_{4d}, \dot{x}_{Ad}, \dot{y}_{Ad}, \theta_{4d}) \) (\( r = 0 \) during the flying phase). Eq. (5), rewritten in the form

\[
D^T (M \ddot{q}_d + d - \ddot{f} - \ddot{r}) = 0
\] (7)

can then be used to verify the correctness of the input data and the used mathematical model. During the flying phase \( (r = 0) \) Eq. (7) is equivalent to the condition that both the body center of mass moves along a parabola (or vertically with acceleration \( g \) pointed downward), and that the total angular momentum of the body parts remains constant. During the support phase \( (r \neq 0) \) Eq. (7) expresses then the condition that the actual time-derivatives of the linear momentum of the system and the angular momentum with respect to the center of mass are due to the actual external forces \( \dddot{f} + \dddot{r} \).

In this way the correctness of the used motion characteristics and the estimated reactions from the trampoline bed as well as the multibody trampolinist model can be verified.

The mass–inertial characteristics of segments of the proposed human body model have been estimated according to Zatsiorsky and Seluyanov (1985), and a back somersault in pike position shown previously in Figs. 3–6 was chosen to demonstrate a solution to the inverse dynamics problem. The computed control variations are reported in Fig. 7. The biggest values as well as variations in control torques can be observed during the support phase (0–0.4 s), while during the flying phase (0.4–1.6 s) the most important for the somersault performance are variations in hip and shoulder control torques. The control torques in elbow and neck also play an essential role.

5. The inverse dynamics simulation (direct dynamics problem)

To solve the inverse dynamics simulation, viz the direct dynamics problem (dynamic simulation) in which the

---

**Fig. 6.** The reactions from the trampoline bed during the support phase.
calculated control is used as the input signal, the
dynamic Eq. (2) needs to be rearranged to
\[
\dot{q} = M^{-1}(q)(f(q) - d(q, \dot{q}) + r(q) + B^T \tau_d(t)), \tag{8}
\]
where \(\tau_d(t)\) is a continuous time function fitting the
inverse dynamics discrete solution. The initial state
values \(q_0 = q(t_0)\) and \(\dot{q}_0 = \dot{q}(t_0)\) should match those used
as input data to the inverse dynamics solution at \(t = t_0\).
A fourth-order adaptive step size Runge–Kutta inte-
grator was used to produce the dynamic simulation with
\(\tau_d(t)\) as the control.

By definition, if \(\tau_d(t)\) are calculated precisely enough,
the solution \(q(t)\) and \(\dot{q}(t)\) to Eq. (8) should match \(q_d(t)\)
and \(\dot{q}_d(t)\). In practice the numerically exact consistency
can usually be achieved only at the beginning of
simulation, provided that \(q_0 = q_d(t_0)\) and \(\dot{q}_0 = \dot{q}_d(t_0)\).
For \(t > t_0\), and especially for long simulation times,
a difference between the simulated and measured state
variables is usually observed, and the difference may
tend to increase in time. There are many reasons for this
inverse dynamics simulation failure phenomenon; see also
Kuo (1998), Risher et al. (1997), Tashman et al. (1995),
and Zajac (1993). The integration truncation errors
seem to be the most obvious cause but, in the authors’
option, they are of minor importance. The other source
of the simulation failure is an inaccuracy in determining
\(q_d(t)\), \(\dot{q}_d(t)\) and \(\ddot{q}_d(t)\), and during the support phase,
\(x_{Ad}(t)\), \(y_{Ad}(t)\) and \(\theta_{Ad}(t)\). The inaccuracy is involved in
the mapping from the measured (*) to continuous (d)
data, and in the method of determining \(\dot{q}_d(t)\) and \(\ddot{q}_d(t)\)
from \(q_d(t)\). The calculated control variations \(\tau_d(t)\) are
then influenced by these errors. By adequate preparation
of the recorded data this source of inverse dynamics
simulation failure can, however, be minimized or even
avoided as argued by Risher et al. (1997).

The inverse dynamics simulation failure is finally
caused by inaccuracy of the used mapping from
the discrete control variations \(\tau^*_d(t)\) obtained from the
inverse dynamics problem to the continuous \(\tau_d(t)\)
variations used in the direct dynamics simulation,
termed by Risher et al. (1997) as an insufficient control
signal dimensionality. Risher et al. stated that the
mappings with insufficient dimensionality (such as linear
interpolation between discrete controls) never assures
the replication of the inverse dynamics simulation
results compared to the data used as input to inverse
dynamics problem. In our study we used cubic spline
approximation/interpolation of sampled solution
\(\tau^*_d(t)\) to produce \(\tau_d(t)\), and achieved good consis-
tency between the results from the inverse dynamics
simulation and the data used as input in the
inverse dynamics problem, especially during the flying
phase.

The comparison between the measured motion and
the inverse dynamics simulation results, for the con-
cidered back somersault in pike position, is illustrated
in Fig. 8, where the animations of the two motions
are demonstrated. The thick gray line shows the
simulated positions in time, and the thin solid black
line stands for the measured positions at the same
sampled times. It can be seen that there is a negligible
difference between the motions. The conformability
was a little spoiled (but was still acceptable) for the
support phase (not reported here). In this case, the
inverse dynamics simulation failure was mainly due to
the limited accuracy of the measured kinematic char-
acteristics \(x_{Ad}(t)\), \(y_{Ad}(t)\) and \(\theta_{Ad}(t)\), and the trampoline

Fig. 7. Solution to the inverse dynamics problem—the torques versus time in the joints.
bed stiffness/damping characteristics reported in Eq. (1). The current research is focused on improving the measurements.

6. Final remarks and conclusions

A mathematical model for the analysis of front and back somersaults on the trampoline was developed. It can be applied to the solution of inverse dynamics problem as well as to the direct dynamics simulation. The same governing equations are used for the flying phase of somersault—when the body rotates above the trampoline bed, and for the support phase—when the athlete touches the bed. The only exception is that in the support phase the reactions from the bed on the feet are involved.

The inverse dynamics problem is solved by using the motion characteristics measured from the actual somersault performances—the body position (in the both phases) and the bed deflections (in the support phase) in time. The applied moments of muscle forces at the joints that result in the given motion are then calculated. By analyzing the results, the nature of the stunts and the way the human body is maneuvered can be studied. This may be a valuable means for supporting the teaching of and mastering the trampoline somersaults.

The torques calculated from the inverse dynamics are then used as control signals to the dynamic simulation. In the flying phase, the obtained results faithfully replicate the data that served as input to the inverse dynamics problem. In the support phase the conformability is a little spoiled but still acceptable. To reduce the inverse dynamics simulation failure in the latter case, the model of interactions between feet and bed needs to be improved and better accuracy of measurements of the bed deflections during the support phase should be achieved. The simulation results provide a way of checking the inverse dynamics procedures as well as searching among different modifications of the control signals that result in coordinated (optimized) movement. The influence of typical control errors on somersault performance can also be studied.

Acknowledgements

The research was supported by the State Committee for Scientific Research, Poland, under Grant 9 T12C 060 17.

Appendix A. The stiffness and damping characteristics of the trampoline

The idea of the measurements is shown in Fig. 9. The trampoline vertical stiffness was measured as the static response $y_A$ to the load $G$ (Fig. 9a). Then, for a given vertical deflection $y_A$ (load $G$), the response $x_A$ to a horizontal force $P_x$ (Fig. 9b) and the response $\theta_4$ to a moment $M_A = G d \cos \theta_4$ (Fig. 9c) were estimated. The measured characteristics are shown in Fig. 10. As it can be seen, in the vertical direction the trampoline behaves as a hardening spring, and the results do not significantly depend on where the load is placed, that is how far from the bed center. The horizontal force $P_x$ shows linear dependency on $x_A$, while its dependency on $y_A$ is negligible. Finally, for a given $y_A$, the dependence of $M_A$ on $\theta_4$ is linear, and the stiffness coefficient $k_\theta$ increases (linearly) in function of $y_A$. The quantitative stiffness characteristics are

\[ G = 6230y_A^2 + 1530y_A, \quad P_x = 240x_A, \]
\[ M_\theta = (22y_A - 0.2)\theta_4, \quad (A.1) \]

where $x_A$, $y_A$ and $\theta_4$ should to be substituted in (m) and (deg) in order to obtain $G$, $P_x$ and $M_\theta$ by (N) and (N m), respectively.
The damping characteristics of the trampoline were estimated as follows. For a given deflection \( y_A \) under the corresponding weight \( G \), the vertical, horizontal and rotational free vibrations of the loaded plate (Fig. 9) had been initiated, and the logarithmic decrements of the vibrations were measured. Then, following the well-known formulae from the linear vibration theory (Hartog, 1968), the damping ratios were determined. For the sake of brevity, the respective expressions and the measurement results are not reported. The estimated damping characteristics are

\[
G = (178y_A^2 + 43.5y_A)\ddot{x}_A, \quad P_x = 660\dot{x}_A, \quad M_A = 1.2\dot{\theta}_A, \tag{A.2}
\]

where \( y_A, \dot{x}_A, \ddot{x}_A \) and \( \dot{\theta}_A \) should to be substituted by (m), (m/s) and (1/) in order to obtain \( G, P_x \) and \( M_A \) in (N) and (N m), respectively.

**Appendix B. The explicit form of dynamic Eq. (2)**

To facilitate the presentation, let us first introduce the following partition of the generalized coordinates \( \mathbf{q} \) and the consequent block form of the generalized mass matrix \( \mathbf{M} \):

\[
\mathbf{q} = \begin{bmatrix} \dot{x}_H, \dot{y}_H, \dot{\varphi}_1 \end{bmatrix}^T, \quad \mathbf{q}_L = \begin{bmatrix} \dot{\varphi}_2, \dot{\varphi}_3, \dot{\varphi}_4 \end{bmatrix}^T, \quad \mathbf{q}_U = \begin{bmatrix} \dot{\varphi}_5, \dot{\varphi}_6 \end{bmatrix}^T, \quad \mathbf{q}_G = \begin{bmatrix} \dot{\varphi}_7 \end{bmatrix},
\]

\[
\mathbf{M} = \begin{bmatrix}
\mathbf{M}_{TT} & \mathbf{M}_{TL} & \mathbf{M}_{TU} & \mathbf{M}_{TG} \\
\mathbf{M}_{LT} & \mathbf{M}_{LL} & 0 & 0 \\
\mathbf{M}_{UT} & 0 & \mathbf{M}_{UU} & 0 \\
\mathbf{M}_{GT} & 0 & 0 & \mathbf{M}_{GG}
\end{bmatrix}, \tag{B.1}
\]

Then, after denoting

\[
m = m_1 + \cdots + m_7, \quad \mu_1 = m_1c_1 + (m_3 + m_6)l + m_7l_1, \quad \mu_2 = m_2c_2 + (m_3 + m_4)l_2, \quad \mu_3 = m_3c_3 + m_4l_3, \quad \mu_4 = m_4c_4, \quad \mu_5 = m_5c_5 + m_6l_5, \quad \mu_6 = m_6c_6, \quad \mu_7 = m_7c_7,
\]

\[
J_{H1} = J_{e1} + m_1l_1^2, \quad J_{S5} = J_{e5} + m_5l_5^2, \quad J_{H2} = J_{e2} + m_2l_2^2, \quad J_{S6} = J_{e6} + m_6l_6^2, \quad J_{H3} = J_{e3} + m_3l_3^2 + m_5l_5^2, \quad J_{W7} = J_{e7} + m_7l_7^2, \quad J_{A4} = J_{e4} + m_4c_4.
\]
the blocks of matrix $\mathbf{M}$ are:

$$
\mathbf{M}_{TT} = \begin{bmatrix}
m & 0 & -\mu_1 \cos \phi_1 \\
\times m & -\mu_1 \cos \phi_1 \\
\times J'_{H1} & & \\
\end{bmatrix},
$$

$$
\mathbf{M}_{TL} = \begin{bmatrix}
\mu_2 \cos \phi_2 & \mu_3 \cos \phi_3 & \mu_4 \cos \phi_4 \\
\mu_2 \sin \phi_2 & \mu_3 \sin \phi_3 & \mu_4 \sin \phi_4 \\
0 & 0 & 0 \\
\end{bmatrix},
$$

$$
\mathbf{M}_{TU} = \begin{bmatrix}
\mu_5 \cos \phi_5 & \mu_6 \cos \phi_6 \\
\mu_5 \sin \phi_5 & \mu_6 \sin \phi_6 \\
-\mu_2 l_s \cos (\phi_5 - \phi_1) & -\mu_2 l_s \cos (\phi_6 - \phi_1) \\
\end{bmatrix},
$$

$$
\mathbf{M}_{TG} = \begin{bmatrix}
-\mu_7 \cos \phi_7 \\
-\mu_7 \sin \phi_7 \\
\mu_7 l_1 \cos (\phi_7 - \phi_1) \\
\end{bmatrix},
$$

$$
\mathbf{M}_{LL} = \begin{bmatrix}
J'_{H2} & \mu_3 l_2 \cos (\phi_3 - \phi_2) & \mu_4 l_2 \cos (\phi_4 - \phi_2) \\
\times J'_{K3} & \mu_5 l_3 \cos (\phi_5 - \phi_3) \\
\times \times J'_{A4} & & \\
\end{bmatrix},
$$

$$
\mathbf{M}_{UU} = \begin{bmatrix}
J'_{S5} & \mu_6 l_5 \cos (\phi_6 - \phi_5) \\
\times J'_{E6} & & \\
\end{bmatrix},
$$

$$
\mathbf{M}_{GG} = [J'_{W7}],
$$

where $\times$ denote the symmetric entries, $m_i$, $J_{ci}$, $l_i$, and $c_j$ ($i = 1, \ldots, 7$) which are, respectively, the masses of the body segments as seen in Fig. 1, their mass moments of inertia with respect to the center of mass, their lengths, and the distances from the joints H, K, A, S, E and W to the appropriate mass centers, and $l_s$ is the distance between the joints H and S.

The other components of Eq. (2) are

$$
d = \begin{bmatrix}
\mu_1 \phi_1^2 \sin \phi_1 - \sum_{i=2}^{6} \mu_1 \phi_i^2 \sin \phi_i + \mu_2 \phi_2^2 \sin \phi_2 \\
-\mu_1 \phi_1^2 \cos \phi_1 + \sum_{i=2}^{6} \mu_1 \phi_i^2 \sin \phi_i - \mu_2 \phi_2^2 \cos \phi_2 \\
\mu_2 l_s \phi_2^2 \sin (\phi_1 - \phi_2) + \mu_3 l_3 \phi_3^2 \sin (\phi_2 - \phi_3) - \mu_3 l_2 \phi_3^2 \sin (\phi_3 - \phi_2) - \mu_4 l_2 \phi_4^2 \sin (\phi_4 - \phi_2) - \mu_4 l_3 \phi_4^2 \sin (\phi_4 - \phi_3) \\
\mu_4 l_2 \phi_2^2 \sin (\phi_4 - \phi_2) + \mu_4 l_3 \phi_3^2 \sin (\phi_4 - \phi_3) - \mu_4 l_5 \phi_5^2 \sin (\phi_5 - \phi_3) - \mu_5 l_5 \phi_5^2 \sin (\phi_5 - \phi_2) - \mu_6 l_5 \phi_5^2 \sin (\phi_5 - \phi_4) \\
-\mu_6 l_5 \phi_6^2 \sin (\phi_5 - \phi_4) + \mu_6 l_5 \phi_6^2 \sin (\phi_6 - \phi_5) - \mu_6 l_5 \phi_6^2 \sin (\phi_6 - \phi_4) \\
\mu_7 l_1 \phi_7^2 \sin (\phi_7 - \phi_1) \\
\end{bmatrix},
$$

$$
f = \begin{bmatrix}
0 \\
-\mu_1 \phi_1 \sin \phi_1 \\
-\mu_1 \phi_1 \sin \phi_2 \\
-\mu_2 \phi_2 \sin \phi_3 \\
-\mu_2 \phi_2 \sin \phi_4 \\
-\mu_2 \phi_2 \sin \phi_5 \\
-\mu_2 \phi_2 \sin \phi_6 \\
-\mu_2 \phi_2 \sin \phi_7 \\
\end{bmatrix},
$$

$$
r = \begin{bmatrix}
-R_c \\
R_f \\
-R_s l_2 \cos \phi_2 - R_s l_2 \sin \phi_2 \\
-R_s l_3 \cos \phi_3 + R_s l_3 \sin \phi_3 \\
M_A \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix},
$$

$$
\mathbf{B}^T \tau = \begin{bmatrix}
\tau_1 \\
\tau_2 \\
\tau_3 \\
\tau_4 \\
\tau_5 \\
\tau_6 \\
\end{bmatrix},
$$

$$
\mathbf{T} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.
References