Session 6

Vector Analysis: Gradient, Divergence, Curl

1 Introduction

Concepts from vector analysis are useful when investigating functions of more than one variable. Vector operators such as the divergence or the curl are physically meaningful and widely employed in applications. This session reviews the basics of vector analysis.

A scalar is a quantity $a$ characterized by its magnitude. Specification of a vector $\mathbf{v}$ requires stating its direction as well as its magnitude $|\mathbf{v}| = v$. Vectors $\mathbf{u}$ and $\mathbf{v}$ can be added

$$\mathbf{w} = \mathbf{u} + \mathbf{v}$$

Vector addition is commutative and associative. Vectors can also be multiplied by scalars. Unit vectors are vectors of unit length while the zero vector has zero length and arbitrary direction. A set of linearly independent unit vectors in 3D Euclidean space are called unit vectors. In Cartesian coordinates these are written as $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. Any other vector can then be expressed by stating its scalar components, $v_x = v \cos \alpha$, $v_y = v \cos \beta$ and $v_z = v \cos \gamma$, (the projections of the vector on the directions of the unit vectors), i.e.

$$\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$$

Here $\alpha$, $\beta$ and $\gamma$ are the direction cosines of $\mathbf{v}$ while the magnitude of $\mathbf{v}$ is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

The direction cosines satisfy the equation

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = l^2 + m^2 + n^2 = 1$$

and the unit vector in the direction of $\mathbf{v}$, $\mathbf{v}_1$ is

$$\mathbf{v}_1 = \frac{v_x}{v} \mathbf{i} + \frac{v_y}{v} \mathbf{j} + \frac{v_z}{v} \mathbf{k}$$
2 Scalar, Vector and Multiple Products

The scalar (or dot) product of two vectors \( \mathbf{a} \) and \( \mathbf{b} \) is

\[
\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z = ab \cos \theta
\]

is a scalar quantity. It is equal to the length of \( \mathbf{b} \) times the magnitude of the projection of \( \mathbf{a} \) onto \( \mathbf{b} \) and it is also equal to the length of \( \mathbf{a} \) times the magnitude of the projection of \( \mathbf{b} \) onto \( \mathbf{a} \). Here \( \theta \) is the angle between the vectors.

The dot product is commutative and distributive. Also, the dot product of orthogonal (i.e. linearly independent) vectors is zero. The dot product of a vector with itself is the square of its magnitude.

The vector (or cross) product of two vectors \( \mathbf{a} \) and \( \mathbf{b} \) is

\[
\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y) \mathbf{i} + (a_z b_x - a_x b_z) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k} = \mathbf{c}
\]

This vector \( \mathbf{c} \) is directed perpendicularly to the plane of the two vectors and has the magnitude

\[|\mathbf{c}| = c = |\mathbf{a} \times \mathbf{b}| = ab|\sin \theta|\]

The vector product is not commutative but it is distributive. Also, the vector product of two parallel vectors is zero.

The tangential velocity vector in rotating systems and the moment vector in mechanics are examples of the above.

Three multiple products are important \((\mathbf{a} \cdot \mathbf{b})\mathbf{c}\), the product of the scalar \(ab \cos \theta\) and the vector \(\mathbf{c}\); \((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\), the triple scalar product (and the volume of the parallelepiped formed by the three vectors); and \((\mathbf{a} \times \mathbf{b}) \times \mathbf{c}\), a vector in the plane of \(\mathbf{a}\) and \(\mathbf{b}\), perpendicular \(\mathbf{a}\).\(\mathbf{c}\).

3 Differentiation of Vectors

The derivative of a vector with respect to a parameter \(t\) is defined as

\[
\frac{d\mathbf{v}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t}
\]

If \(\mathbf{v} = f(t)i + g(t)j + h(t)k\) then

\[
\frac{d\mathbf{v}}{dt} = \frac{df}{dt}i + \frac{dg}{dt}j + \frac{dh}{dt}k
\]

Derivative formulae for vector products are readily obtained from the above definition. Also, note that the derivative of a vector of constant length but changing direction is perpendicular to the vector.
4 Space Curves

A curve in 3D space is defined by the parametric equations

\[ x = x(t), \quad y = y(t), \quad z = z(t) \]

The position vector \( \mathbf{r} \) from the origin to a point \( P(x, y, z) \), corresponding to a specific value of \( t \) is

\[ \mathbf{r} = xi + yj + zk \]

As the value of the parameter changes from \( t \) to \( t + \Delta t \) the point \( P \) traverses a space curve. Let \( s \) be the arc length along the curve. In the limit when \( \Delta t \to 0 \),

\[
\frac{d\mathbf{r}}{dt} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} = \left( \frac{dx}{ds} \frac{ds}{dt} \mathbf{i} + \frac{dy}{ds} \frac{ds}{dt} \mathbf{j} + \frac{dz}{ds} \frac{ds}{dt} \mathbf{k} \right) = \frac{d\mathbf{r}}{ds} = \frac{u}{ds} \]

\( u \) is the unit tangent vector to the curve and pointing in the direction of increasing arc length, therefore

\[
\frac{ds}{dt} = \sqrt{\left( \frac{dx}{dt}\right)^2 + \left( \frac{dy}{dt}\right)^2 + \left( \frac{dz}{dt}\right)^2}
\]

where \( ds \) is the element of arc length of the space curve.

Further, since \( u \) is a unit vector, the rate of change of \( u \) with \( s \),

\[
\frac{du}{ds} = \frac{d^2\mathbf{r}}{ds^2} = \frac{d^2x}{ds^2} \mathbf{i} + \frac{d^2y}{ds^2} \mathbf{j} + \frac{d^2z}{ds^2} \mathbf{k} = \frac{d\mathbf{u}/dt}{ds/dt}
\]

is a vector perpendicular to the tangent vector. The length of this new vector is

\[
\left| \frac{du}{ds} \right| = \sqrt{\left( \frac{d^2x}{ds^2} \right)^2 + \left( \frac{d^2y}{ds^2} \right)^2 + \left( \frac{d^2z}{ds^2} \right)^2}
\]

This is called the curvature of the curve and it measures the rate of change of direction of the curve with distance along it. The reciprocal of the curvature is the radius of curvature \( \rho \). Introducing the principal unit normal vector

\[
\mathbf{n} = \frac{\frac{du}{ds}}{|\frac{du}{ds}|} = \frac{du}{ds} \rho
\]

yields

\[
\frac{du}{ds} = \frac{n}{\rho} = n \sqrt{\left( \frac{d^2x}{ds^2} \right)^2 + \left( \frac{d^2y}{ds^2} \right)^2 + \left( \frac{d^2z}{ds^2} \right)^2}
\]
The two vectors \( \mathbf{u} \) and \( \mathbf{n} \) define a third vector \( \mathbf{b} = \mathbf{u} \times \mathbf{n} \) called the binormal vector. Differentiating with respect to \( s \)

\[
\frac{d\mathbf{b}}{ds} = \rho^2 \begin{vmatrix}
dx \ dy \ dz & dx \ dy \ dz & dx \ dy \ dz \\
\frac{dx}{ds} \ rac{dy}{ds} \ rac{dz}{ds} & \frac{dx}{ds} \ rac{dy}{ds} \ rac{dz}{ds} & \frac{dx}{ds} \ rac{dy}{ds} \ rac{dz}{ds} \\
\frac{dx}{ds^3} \ rac{dy}{ds^3} \ rac{dz}{ds^3} & \frac{dx}{ds^3} \ rac{dy}{ds^3} \ rac{dz}{ds^3} & \frac{dx}{ds^3} \ rac{dy}{ds^3} \ rac{dz}{ds^3}
\end{vmatrix}
\mathbf{n} = -\frac{1}{\tau}
\]

where \( 1/\tau \) is the torsion and \( |\tau| \) is the radius of torsion of the curve. Finally

\[
\frac{d\mathbf{n}}{ds} = \frac{1}{\tau} \mathbf{b} - \frac{1}{\rho} \mathbf{u}
\]

The above set of differential relations involving the three mutually perpendicular unit vectors associated with the curve are called Frenet’s formulae.

Specific instances of the above occur in problems in kinematics of points and rigid bodies.

5 The \( \nabla \) Operator: Gradient, Divergence and Curl

Consider a scalar function of position \( \psi(x, y, z) \). The gradient vector of \( \psi \) is

\[
\text{grad } \psi = \nabla \psi = i \frac{\partial \psi}{\partial x} + j \frac{\partial \psi}{\partial y} + k \frac{\partial \psi}{\partial z}
\]

So, the component of \( \nabla \psi \) in any direction is the rate of change of \( \psi \) with respect to distance along that direction and \( \text{grad } \psi \) is a vector function associated with \( \psi \) in a form which is independent of the system of coordinates being used.

The dot product of the tangent unit vector to a curve \( \mathbf{u} = \frac{d\mathbf{r}}{ds} \) and \( \nabla \psi \) is

\[
\mathbf{u} \cdot \nabla \psi = \frac{d\mathbf{r}}{ds} \cdot \nabla \psi = \frac{d\psi}{ds}
\]

Therefore, \( \nabla \psi \) points in the direction of greatest rate of change of \( \psi \) and its length is equal to the value of that maximum derivative.

It is often convenient to think of \( \nabla \) as a vector operator, i.e.

\[
\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}
\]

Consider now the vector function of position \( \mathbf{F}(x, y, z) = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} \). The scalar and vector products of the operator \( \nabla \) into \( \mathbf{F} \) are

\[
\nabla \cdot \mathbf{F} = i \cdot \frac{\partial F}{\partial x} + j \cdot \frac{\partial F}{\partial y} + k \cdot \frac{\partial F}{\partial z} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \text{div } \mathbf{F}
\]
which is called the divergence of F and

\[ \nabla \times F = i \times \frac{\partial F}{\partial x} + j \times \frac{\partial F}{\partial y} + k \times \frac{\partial F}{\partial z} = \]

\[ i \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + j \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + k \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) = \text{curl } F \]

which is called the curl of F.

The derivative of F in the direction of a unit vector \( \mathbf{u} \) is

\[ \frac{dF}{ds} = (\mathbf{u} \cdot \nabla) F \]

where

\[ \mathbf{u} \cdot \nabla = u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z} \]

The following is a collection of useful differentiation formulae.

\[ \nabla \cdot \psi \mathbf{u} = \psi \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \psi \]

\[ \nabla \times \psi \mathbf{u} = \psi \nabla \times \mathbf{u} + \nabla \psi \times \mathbf{u} \]

\[ \nabla \cdot \mathbf{u} \times \mathbf{v} = \mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times \mathbf{v} \]

\[ \nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{u} (\nabla \cdot \mathbf{v}) - \mathbf{v} (\nabla \cdot \mathbf{u}) \]

\[ \nabla (\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}) \]

\[ \nabla \times (\nabla \psi) = \text{curl } (\text{grad } \psi) = 0 \]

\[ \nabla \cdot (\nabla \times \mathbf{u}) = \text{div } (\text{curl } \mathbf{u}) = 0 \]

\[ \nabla \cdot (\nabla \psi_1 \times \nabla \psi_2) = 0 \]

\[ \nabla \times (\nabla \times \mathbf{u}) = \text{curl } (\text{curl } \mathbf{u}) = \text{grad } (\text{div } \mathbf{u}) - \nabla^2 \mathbf{u} \]

where

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]
is the Laplacian operator and
\[
\nabla^2 u = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (u_x i + u_y j + u_z k) = \\
\left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) i + \left( \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2} \right) j + \left( \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right) k
\]

Finally, for the position vector \( \mathbf{r} \) the following are valid
\[
\nabla \cdot \mathbf{r} = 3 \\
\nabla \times \mathbf{r} = 0 \\
\mathbf{u} \cdot \nabla \mathbf{r} = \mathbf{u}
\]

where \( \mathbf{u} \) is any vector.

6 Practice Problems

6.1

The temperature at a point in space is \( T = xy + yz + zx \).

a) Find the direction in which the temperature changes most rapidly with distance from (1, 1, 1). What is the maximum rate of change?

b) Find the derivative of \( T \) in the direction of the vector \( 3i - 4k \) at (1, 1, 1).

Answer: a) Here \( \nabla T = (y + z)i + (x + z)j + (y + x)k \). The maximum rate of change at (1, 1, 1) is \( |\nabla T(1, 1, 1)| = 2\sqrt{3} \) and direction cosines are
\[
\frac{\nabla T}{|\nabla T|} = \frac{1}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j + \frac{1}{\sqrt{3}} k = \cos \alpha i + \cos \beta j + \cos \gamma k
\]

b) The required derivative is
\[
\nabla T(1, 1, 1) \cdot \frac{3i - 4k}{|3i - 4k|} = -\frac{2}{5}
\]
6.2

For each of the following vector functions $\mathbf{F}$, determine whether $\nabla \phi = \mathbf{F}$ has a solution and determine it if it exists.

a) $\mathbf{F} = 2xyz^3\mathbf{i} - (x^2z^3 + 2y)\mathbf{j} + 3x^2yz^2\mathbf{k}$

b) $\mathbf{F} = 2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 1)\mathbf{k}$

Answer:

a) Here $\nabla \phi = \mathbf{F}$ requires $\nabla \times \mathbf{F} = 0$ which is not the case here, so no solution.

b) Here $\nabla \times \mathbf{F} = 0$ so that

$$\phi(x, y, z) = x^2y + y^2z + z + c$$