A Method for Use of Cyclic Symmetry Properties in Analysis of Nonlinear Multiharmonic Vibrations of Bladed Disks

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Introduction

It is well known that in analysis of linear vibration, use of cyclically symmetric properties of bladed disks allows the analysis of a whole bladed-disk assembly to be reduced to analysis of its one sector, which usually comprises a blade, a disk sector and, possibly, parts of shrouds adjoined to the blade. Methods for use of the cyclic symmetry properties of linear mechanical structures in analysis of natural frequencies and mode shapes, and in forced response analysis, have been developed more than 30 years. Many papers on the subject including papers [1–3], together provide a theoretical basis and methods for analysis of linear vibrations of cyclically symmetric structures. It has been shown that cyclic symmetry allows a sector model to be used instead of a whole bladed disk model, which decreases number of degrees-of-freedom in the resulting equations by a factor equal to the number of blades in a bladed disk, without any compromise with modeling accuracy. Coupling of vibrations of all sectors can be accounted for by special boundary conditions imposed on the interfaces where a sector considered interacts with neighboring ones. This method of taking into account cyclic symmetry, for linear systems, has become conventional and numerous applications in analysis of linear vibrations of bladed disks and other structures have been developed. An option allowing use of cyclic symmetry has become a standard in the majority of commercial finite element programs.

In contrast to the case of linear systems for non-linear systems, up to date, there has been no a rigorously formulated and universal method available in the literature that would take advantages of cyclic symmetry for analysis of strongly nonlinear vibrations in a consistent way.

There are very few papers on numerical analysis of strongly nonlinear vibration of cyclically symmetric structures but calculation techniques and investigations provided in papers [4,5] for a models with cubic nonlinear springs should be noticed.

One of first calculations of cyclically symmetric bladed disks with gaps was proposed in [6], where single degree-of-freedom and beam models of blades are explored. Later papers on analysis of bladed disks with underplatform dampers by [7,8] and shrouded bladed disks [9], differ from the above-mentioned paper by modeling the bladed disk and friction forces, but use similar approaches to account for the cyclic symmetry, which are based on assuming that excitation and vibrations are monoharmonic and leaving aside the question about applicability of the method developed for linear structures with cyclic symmetry to a nonlinear structure. Multiharmonic vibrations of bladed disks are analyzed in [10,11] although the method for calculation of the cyclically symmetric bladed disks is not elaborated in either paper.

In this paper an effective method for analysis of steady-state nonlinear vibrations of cyclically symmetric bladed disks subjected to arbitrary distributed in space and periodic in time loads is proposed.

At the beginning the possibility for reduction of calculation of vibrations of a whole bladed disk to analysis of its sector model is rigorously proved and conditions of the method applicability are derived. Types of forcing that can occur in practice of bladed disk analysis and satisfy to the conditions are discussed. Then the nonlinear equations of multiharmonic motion are formulated for a sector model for two cases: (i) using sector finite element matrices and (ii) using sector mode shapes and frequencies. Calculations validating the developed method and a numerical investigation of a realistic high-pressure turbine bladed disk with shrouds have demonstrated the high efficiency of the method. [DOI: 10.1115/1.1644558]

Bladed Disks of Cyclically Symmetric Design: Sector and Order of Cyclic Symmetry. Bladed disks are designed in majority of cases to be cyclically symmetric. Cyclically symmetric bladed disks are such bladed disks for which it is possible to find a rotationally periodic, “cyclic” part of the structure which can form the whole bladed disk by simple rotations of this part relatively to the axis of the symmetry. The rotation angles have to be multiples of $2 \pi / N$ where integer number $N$ is so-called “order of the cyclic symmetry,” which is equal to the number of cyclic parts in the whole structure. This cyclic part is sometimes called also
“sector” reflecting the fact that the cyclic part can be in many cases separated from the whole structure by two planes passing through the symmetry axis with angle $2\pi/N$ between them. The structure possesses all cyclic symmetry properties also for a general case when surfaces separating the cyclic part can have very complex and curved shapes. Because of that word “sector” is used in this paper in a general meaning assuming that there is no restrictions on shapes of the sector boundaries.

As evident all co-factors, $N_j$, of $N=N_1^2N_2^2N_3\ldots$ if such exist, and all possible combinations of multiplication of these co-factors are also orders of cyclic symmetry. In order to take a maximum of advantages of using cyclic symmetry properties the largest possible order of cyclic symmetry is used in analysis. This choice of the cyclic symmetry order provides the smallest sector of the cyclically symmetric bladed disk and accordingly the smallest number of degrees-of-freedom in the sector model. Number of blades that has to be included into one sector depends on a design of a bladed disk. For bladed disks of gas turbine engines a sector can usually include only one blade (as shown in Fig. 1) and number of sectors, $N$, is equal to the number of blades, although there are cases when bladed disks have such design that the sector has to include two or more blades.

**Major Relationships and Applicability Conditions.** If a bladed disk is cyclically symmetric its equations of motion can be written in the following form:

$$Kq + Cq + Mq + f_1(q_{j-1}, q_j) + f_j(q_j, q_{j+1}) = p_j(t)$$

where $j=1, N$ is a sector number; $K$, $C$, and $M$ are conventional linear stiffness, damping and mass matrices of one sector which are the same for all sectors of the cyclically symmetric system; $q_j(t)$ is a vector of displacements of $j$-th sector; $p_j(t)$ is a vector of excitation forces; $f_1(q_{j-1}, q_j)$ and $f_j(q_j, q_{j+1})$ are forces of nonlinear interaction of $j$th sector and sectors adjacent to it from the left and from the right, they are dependent on displacements of the considered $j$th sector and neighboring $(j-1)$th and $(j+1)$th sectors. Due to the cyclic symmetry the functional dependencies of vectors $f_1(q_{j-1}, q_j)$ and $f_j(q_j, q_{j+1})$ on the displacements are the same for all sectors, and, moreover, $f_1(q_{j-1}, q_j)=f_1(q_{j+1}, q_j)$, although different displacements can produce different force values.

Equation (1) represents a system of $N$ matrix equations. In this system of equations subscript values 0 and $N+1$ appearing in $q_{j \pm 1}$ for $j=1$ and $j=N$ are replaced by $N$ and 1 accordingly since first sector interacts with $N$th sector. If arbitrary periodic excitation forces, $p_j(t)$, applied to each $j$th sector are similarly distributed over sector nodes and different only in a fixed phase shift, $\delta t$, between adjacent sectors, then these excitation forces can be written in the form

$$p_j(t) = p(t + (j - 1) \delta t)$$

where $\delta t = \pm T/N$ and $T$ is the period of the force variation, or $\delta t = 0$ when the phase shift between the forces applied to different sectors is 0.

For this kind of excitation and when linear and nonlinear properties of the bladed disk are cyclically symmetric it is possible to write explicit relationship between displacements of all sectors. This relationship has the form

$$q_j(t) = q(t + (j - 1) \delta t)$$

where $q$ is the vector of displacements of first sector, which is also periodic with period $T$. The relationship given by Eq. (3) is the major relationship which allows all forces applied to a sector of the bladed disk including forces of interaction with adjacent sectors to be expressed in terms of its own displacements only. As a result the dynamic equation for a sector can be written in the form that describe vibrations of the whole bladed disk. Validation of this relationship can be proved by substitution of Eqs. (2) and (3) into Eq. (1). As a result of such substitution one obtains a system of $N$ equations of identical form:

$$Kq_1 + Cq_1 + Mq_1 + f_1(q_0, q_1) + f_1(q_{N-1}, q_N) = p_1(t)$$

$$f_j(q_j, q_{j+1}) = p_j(t)$$

where $t = t + (j - 1) \delta t$ and $j$ takes values from 1 to $N$. One can see that these equations differ for different sectors by the phase of the time variable only and the solution found, $q(t)$, for one sector from Eq. (4) satisfies equations for all sectors.

Moreover, comparing in Eq. (4) $f_j$ and $f_1$ allows a relationship for forces of interaction between the considered sector and adjacent to it on left and right boundaries to be written in the following form:

$$f_j(t) = f_1(t + \delta t)$$

This property together with Eq. (3) provides us a basis for calculation of forced response of the whole cyclically symmetric bladed disk using its sector model without introduction of any assumptions and loss of accuracy of the calculations even for the case of nonlinear vibrations with strong nonlinear forces of any kind.

**Types of Forcing That can be Analyzed by the Method.** The condition formulated for excitation forces by Eq. (2) together with the cyclic symmetry of the bladed disk are the only conditions that have to be satisfied to apply the method proposed. Types of the excitation forces that occur in analysis of vibration of bladed disks and satisfy to this condition are briefly discussed in this section.

**Excitation of Traveling Wave Type.** This type of excitation is one of the most customary in analysis of vibrations of bladed disks. It is typical for rotating bladed disks when they are excited by aerodynamic or other forces that travel relatively to the bladed disk due to the bladed disk rotation with constant speed while preserving their spatial distribution. In cylindrical coordinates, $r$, $z$, and $\varphi$, distribution and travelling of these forces can be expressed in the form: $p = p(r, z, \varphi \pm \omega t)$, where sign “+” corresponds to forward travelling wave and sign “−” correspond to backward travelling wave. All forces of this type satisfy to the condition given by Eq. (2). In our method no restrictions on distribution of the forces over a bladed disk are imposed and these forces can be distributed arbitrarily. Being expanded into Fourier series with respect to bladed disk circumferential coordinate, $\varphi$, they can comprise many harmonic components. The rotation speed, $\omega$, determines the period of the excitation: $T = 2\pi/\omega$ and phase shift is determined as $\delta t = \pm T/N$, where sign “+” corre-
spond to the forces travelling forward and sign ‘’+’’ corresponds to the forces travelling backward with respect to the bladed disk.

The conventional engine-order excitation is a particular kind of the travelling wave forcing considered here. For engine-order excitation the forces are distributed along the circumferential direction of the bladed disk as a sine or cosine function. A number of waves along the circumference is prescribed by the engine order.

**Excitation That is Stationary in Space.** This type of excitation is characteristic for analysis of bladed disk vibration when the bladed disk is immobile, as in static test rigs with shakers attached to the bladed disk, although it can also occur in rotating bladed disks under a special kind of excitation when the excitation forces are moving together with the rotating bladed disk. The stationary excitation forces can be expressed in the form \( p = p_y(r, z, \varphi) p_y(t) \) where \( p_y(r, z, \varphi) \) is a function of spatial distribution of the forces and \( p_y(t) \) is a periodic function of the force variation in time. The conditions of the method applicability can be satisfied in the following cases.

1. A case when distribution of forces is the same for each of all sectors of the bladed disk, i.e., when the function of the spatial distribution, \( p_y(r, z, \varphi) \), is periodic with the period equal to a sector angle, \( T = 2\pi/N_s \):

   \[
   p_y(r, z, \varphi) = p_y(r, z, \varphi + T) = p_y(r, z, \varphi + T). \tag{6}
   \]

   For this case Eq. (2) is satisfied and phase shift between forces applied to different sectors is zero, i.e., \( \theta = 0 \).

2. The method can be also applied for a more complex case, when the function of the spatial distribution can differ for several sectors. However, to apply the method for this case a period for this spatial distribution has to exist. This period, \( T \), can stretch over a group that comprises several sectors, i.e., \( T = n_s 2\pi/N \) where \( n_s \), is the number of sectors in the group. In order to satisfy Eq. (2) the group of \( n_s \) sectors has to be considered as a new “super” sector. Number of such “super” sectors in the whole system is determined as \( N_s = N/n_s \). This allows this case to be reduced to the case considered in the previous paragraph although with smaller magnitude of the order of the cyclic symmetry.

**Multiharmonic Formulation of Equations of Motion Using a Sector Model**

Multiharmonic Expansion of Displacements. In cases where the excitation forces are periodic, it is usually desirable to find steady-state, periodic regimes of response variation. For a search of the periodic vibration response the variation of all degrees-of-freedom (DOF) of the system in time can be represented as a restricted Fourier series, which can contain as many and such harmonic components as are necessary to approximate the sought solution, i.e.,

\[
q(t) = Q_0 + \sum_{j=1}^{n} \left( Q^{(c)}_j \cos m_j \omega t + Q^{(s)}_j \sin m_j \omega t \right) \tag{7}
\]

where \( Q^{(c)}_j \) and \( Q^{(s)}_j \) \((j = 1 \ldots n)\) are vectors of harmonic coefficients for system DOFs upon cosine and sine components marked by superscripts \(^{(c)}\) and \(^{(s)}\) accordingly; \( Q_0 \) is a vector of constant components of the displacements; \( m_j (j = 1 \ldots n) \) are specific numbers of harmonics that are kept in the displacement expansion in addition to the constant component. Equation (7) can be rewritten in more concise and convenient for our derivation form:

\[
q(\tau) = (H^T \otimes I) \mathbf{Q} \tag{8}
\]

where \( H = [1, \cos m_1 \tau, \sin m_1 \tau, \ldots, \cos m_n \tau, \sin m_n \tau]^T \) is a vector of harmonic functions used in the multiharmonic expansion; \( \mathbf{Q} = [Q_0, Q^{(c)}_1, Q^{(s)}_1, \ldots, Q^{(c)}_n, Q^{(s)}_n]^T \) is a total vector including all harmonic coefficients for the sector DOFs; \( \tau = \omega t \) is dimensionless time, \( I \) is identity matrix of size equal to the number of DOFs in the sector; and \( \otimes \) is a symbol of the Kronecker matrix product operator. [12]. This operator maps here vector \( H \) and matrix \( I \) into the following rectangular matrix:

\[
H^T \otimes I = [I, \cos m_1 \tau I, \sin m_1 \tau I, \ldots, \cos m_n \tau I, \sin m_n \tau I] \tag{9}
\]

where \( N_q \) is size of vector \( q \).

**Formulation Using a Finite Element Sector Model.** In accordance with the multi-harmonic balance method, the expansion from Eq. (7) is substituted into the equation of motion (4), after this Eq. (4) is sequentially multiplied by \( \cos m_j \omega t \) and \( \sin m_j \omega t \) for all harmonics from the expansion and integrals over the vibration period, \( T \), are calculated. As a result, the Eq. (4) of motion in time domain is transformed into a frequency domain equation in the following form:

\[
Z(\omega)Q + F(Q) = P \tag{10}
\]

where \( F(Q) = \{F_0(Q), F^{(c)}_1(Q), F^{(s)}_1(Q), \ldots, F^{(c)}_n(Q), F^{(s)}_n(Q)\}^T \) is a vector of harmonic components of nonlinear forces; \( P = \{P_0, P^{(c)}_1, P^{(s)}_1, \ldots, P^{(c)}_n, P^{(s)}_n\}^T \) is a vector of harmonic components of the excitation forces, and \( Z(\omega) \) is the dynamic stiffness matrix of the linear part of the system, constructed for all harmonic components, i.e.,

\[
Z = \text{diag}[Z_1, Z_1, \ldots, Z_n] \tag{11}
\]

where

\[
Z_0 = K; \quad \text{and} \quad Z_j = \begin{bmatrix} K - (m_j \omega)^2 M & m_j \omega C \\ -m_j \omega C & K - (m_j \omega)^2 M \end{bmatrix} \tag{12}
\]

This equation is nonlinear with respect to the harmonic components of the displacements, \( Q \).

In order to take into account conditions imposed on the sector of cyclically symmetric bladed disk by Eqs. (3) and (5) one can partition the vector of sector displacements into three vectors: (i) a vector of displacements at nodes located at the left boundary of the sector, \( q_l \); (ii) a vector of internal displacements, \( q_i \), and (iii) a vector of displacements at nodes located at the right boundary of the sector, \( q_r \).

The major relationship, Eq. (3), imposes constraints on displacements of the nodes located at left, \( q_l \), and right, \( q_r \), boundaries of the sector where this sector interacts with adjacent ones.

\[
\mathbf{q}_l(\tau) = \mathbf{q}_l(\tau + \alpha) \tag{13}
\]

where \( \alpha = \pm 2 \pi/N \) and sign “+” or “−” is chosen here accordingly to the direction of the rotation of the excitation forces in Eq. (2).

These constraints can be formulated with respect to harmonic coefficients for displacements at the right, \( Q_r \), and left, \( Q_l \), boundaries with the help of Eq. (8) in the form

\[
(H^T(\tau + \alpha) \otimes I) \mathbf{Q}_r = (H^T(\tau + \alpha) \otimes I) \mathbf{Q}_l \tag{14}
\]

The vectors of the multiharmonic expansion, \( H(\tau) \), with a constant phase shift, \( \alpha \), are related as

\[
H(\tau + \alpha) = TH(\tau) \tag{15}
\]

where matrix of the transformation, \( T \), takes the following form:

\[
T = \text{diag}[t_0, t_1, \ldots, t_n] \tag{16}
\]

where

\[
t_0 = 1 \quad \text{and} \quad t_j = \begin{bmatrix} \cos m_j \alpha & -\sin m_j \alpha \\ \sin m_j \alpha & \cos m_j \alpha \end{bmatrix} \quad \text{for} \quad j = 1 \ldots n.
\]

From Eq. (14) and Eq. (15) the following relationship between harmonic coefficients for the displacements at right and left boundaries can be derived:
\[ Q_j = (T \otimes I)Q_j, \]  
\( Q_j \) is the displacement vector of the \( j \)th harmonic of the sector.

Selecting from Eq. (17) expressions for each \( j \)th harmonic of the displacements one can write a relationship between coefficients of the multiharmonic expansion at left and right sector boundaries:

\[
\begin{bmatrix} Q_j^{(c)} \\ Q_j^{(s)} \end{bmatrix} = \begin{bmatrix} \cos m_j \alpha I & -\sin m_j \alpha I \\ \sin m_j \alpha I & \cos m_j \alpha I \end{bmatrix} \begin{bmatrix} Q_j^{(c)} \\ Q_j^{(s)} \end{bmatrix},
\]

and then harmonic coefficients of all sector displacements can be expressed through displacements at internal and left boundary nodes in the form

\[
G_j \begin{bmatrix} Q_j^{(c)} \\ Q_j^{(s)} \end{bmatrix} = \begin{bmatrix} \cos \theta_j I & -\sin \theta_j I \\ \sin \theta_j I & \cos \theta_j I \end{bmatrix} \begin{bmatrix} Q_j^{(c)} \\ Q_j^{(s)} \end{bmatrix} = \begin{bmatrix} Q_j^{(c)} \\ Q_j^{(s)} \end{bmatrix}.
\]  
\( \theta_j \) is a sector FRF matrix determined for \( j \)th harmonic.

Equation (20) can now be rewritten, with taking into account the block diagonal structure of the matrix \( Z \), separately for each \( j \)th harmonics in the following form:

\[
Z_j Q_j + \mathcal{F}_j(Q_j) = \mathcal{P}_j; \quad (j = 0, \pi).
\]

It should be noted that vector of \( j \)th harmonics of nonlinear forces, \( \mathcal{F}_j(Q_j) \), is dependent on all multiharmonic components of displacements, \( Q_j \), because of that all harmonics components of Eq. (23) are independent.

The expression given by Eq. (19) for all sector displacements through internal displacements and displacements of nodes located at left boundary is transformed into the following form:

\[
\begin{bmatrix} Q_j^{(c)} \\ Q_j^{(s)} \end{bmatrix} = \begin{bmatrix} G_j^{(c)} \\ G_j^{(s)} \end{bmatrix}.
\]

Matrix of this transformation, \( G_j \), allows Eq. (20) to be rewritten for each \( j \)th harmonic in the form:

\[
E_j^{(FE)}(Q) = Z_j Q_j + \mathcal{F}_j(Q) - \mathcal{P}_j = 0; \quad (j = 0, \pi)
\]

where \( Z_j^{(E)} = G^T_j Z_j G_j; \mathcal{F}_j(Q) = G^T_j \mathcal{F}(Q); \mathcal{P}_j = G^T_j \mathcal{P}; \) \( G = \text{diag}(G_0, G_1, \ldots, G_n) \) and \( G_j \) is a matrix used in Eq. (19) for expression of all sector DOFs through internal and left boundary DOFs for each harmonic.

Equation (20) represents a nonlinear set of equations with respect to \( \mathcal{E}_j = [Q_j^{(c)}, Q_j^{(s)}]^T \) which describes vibration of a whole bladed disk exactly using a finite element model for one sector only.

**Formulation With Use of Complex Arithmetic.** Special structure of matrix, \( Z \), allows the computational efforts necessary for evaluation of the vector of residuals, \( \mathcal{E}(Q) \) to be reduced. This is achieved by formulating with a use of complex numbers, which allows all relationships to be simplified and, moreover, size of the matrices to be reduced by a factor of two. In order to make such formulation the complex vectors for each \( j \)th harmonics of displacements, \( V_j \), nonlinear forces, \( F_j \), and excitation forces, \( P_j \), are introduced in the following form:

\[
Q_j = \mathcal{Q}_j^{(c)} + i\mathcal{Q}_j^{(s)}; \quad P_j = \mathcal{P}_j^{(c)} + i\mathcal{P}_j^{(s)}; \quad \mathcal{F}_j = \mathcal{F}_j^{(c)} + i\mathcal{F}_j^{(s)},
\]

where \( i = \sqrt{-1} \) and all complex quantities in order to differ them from their real counterparts are written here and further in the paper using a different font. A matrix of complex dynamic stiffness for each \( j \)th harmonics, \( \mathcal{Z}_j \), is introduced as

\[
\mathcal{Z}_j = [K - (m_j \omega)^2 M] - im_j \omega C.
\]
In order to exclude linear DOFs Eq. (27) can be rewritten in the form where vector \( \overline{Q}_i \) is partitioned into a vector of nonlinear DOFs, \( \overline{Q}_{in} \), and a vector of linear DOFs, \( \overline{Q}_{lin} \):

\[
\mathcal{E}^{\text{FRF}}(\overline{Q}) = \overline{Q}_{lin}^p + \begin{bmatrix} 0 \\ \overline{\alpha}_l \overline{T}_j(\overline{Q}_{lin}) \end{bmatrix} - \overline{A}_l \overline{P}_j = 0
\]  

(28)

where \( \overline{A}_l \) is a minor of matrix \( \overline{A}_l \) corresponding to nonlinear DOFs. Selecting from Eq. (28) equations corresponding to nonlinear DOFs we obtain the sought for equation of significantly reduced size:

\[
\mathcal{E}_j^{\text{FRF}}(\overline{Q}_{lin}) = \overline{Q}_{lin}^p + \overline{A}_{lin} \overline{P}_j(\overline{Q}_{lin}) - \overline{\varphi}_j \overline{P}_j = 0
\]

(29)

where \( \overline{Q}_{lin} = (\overline{Q}_{lin}^1, \overline{Q}_{lin}^2, \ldots, \overline{Q}_{lin}^{30}) \) is a vector comprising all nonlinear harmonic coefficients. \( \overline{Q}_{lin} = (\overline{\varphi}_j) \) is a vector of complex amplitudes determined for nonlinear DOFs of the bladed disk excited by \( j \)-th harmonic of the excitation forces. These amplitudes are determined for a completely linear system when the nonlinear forces appearing at contact interfaces of the bladed disk are not taken into account. The vector of excitation forces, \( \overline{P}_j \), can include forces applied to all DOFs of the structure, as linear and as nonlinear. FRF matrix including all DOFs, \( \overline{A}_l \), is used to evaluate vector \( \overline{A}_l \overline{P}_j \) and then components corresponding to nonlinear DOFs are selected from the resulting vector to form \( \overline{Q}_{lin}^p \).

Calculation of Sector Frequency Response Function Matrices. The sector FRF matrix can be efficiently generated from natural frequencies, \( \omega_{r_m} \), and mode shapes, \( \phi_{r_m} \), calculated for the sector for a harmonic number needed, \( m_j \), in order to avoid very numerically inefficient operation of the matrix inverse

\[
\overline{A}_l = \overline{Z}^{-1}_j(m_j \omega) = \sum_{r=1}^{N_m} \frac{\phi_{r_m} \phi_{r_m}^*}{(1 - i \eta_{r_m}) \omega_{r_m}^2 - (m_j \omega)^2}
\]

(30)

where subscript “r” is a number of the mode shape in a family of modes with \( m_j \) circumferential waves of displacements; \( \eta_{r_m} \) is damping loss factor determined for \( r \)-th mode of the family with \( m_j \) waves, and \( N_m \) is the number of modes that are used in the modal expansion. In many cases small numbers of modes, \( N_m \), kept in the expansion provides sufficient accuracy in the FRF matrix calculation (analysis of the accuracy obtained for FRF matrices of bladed disks models is performed in [13]).

It should be noted that Eq. (30) is exact when all sector modes of the family of modes with \( m_j \) waves are included in the expansion and when the damping matrix, \( \overline{C} \), represents ‘proportional’ damping (i.e., it can be expressed as a linear combination of the stiffness matrix and mass matrix).

The following form of presentation for the damping matrix describes as structural, frequency-independent damping and as viscous damping:

\[
\overline{C}_j = \mu_1 \frac{1}{m_j \omega} \overline{K} + \mu_2 \overline{M}
\]

(31)

where first summand describes structural, hysteresis damping and two other summands correspond to damping of viscous type; here \( \mu_k (k = 1, 2, 3) \) are coefficients that can be determined theoretically or experimentally. The modal damping loss factors can be then expressed through these coefficients in the form:

\[
\eta_{r_m} = \mu_1 + m_j \omega \mu_2 + m_j^2 \omega^2 \mu_3.
\]

(32)

Assuming here \( \mu_2 = \mu_3 = 0 \) one can describe pure hysteresis damping, and when \( \mu_1 = 0 \) the damping is pure viscous. In many cases the modal damping loss factors, \( \eta_{r_m} \), are determined directly from experiments for each modes included into Eq. (30). The expansion given by Eq. (30) can be accurate enough in many practical cases even when damping is not “proportional” but level of linear damping described by matrix \( \overline{C} \) and interaction between different modes caused by this damping are small.

The vector of complex amplitudes, \( \overline{Q}_{lin} \), introduced in Eq. (29) can be very efficiently calculated as

\[
\overline{Q}_{lin} = \sum_{j=1}^{N_m} \frac{\phi_{r_m} \overline{P}_j}{1 - i \eta_{r_m} \omega_{r_m}^2 - (m_j \omega)^2}
\]

(33)

Multiplication of large vectors, \( \phi_{r_m} \overline{P}_j \), involving all sector DOFs provides a single coefficient for each mode shape, which is usually called “a modal force.” These modal forces take into account arbitrary distribution of the excitation forces over all nodes of the sector model. For a case when distribution of the excitation forces over a sector is preserved for all rotation frequencies additional decrease of computational efforts can be achieved since for this case calculation of the modal forces involving many degrees-of-freedom is carried out only one time. At the same time for this case values of the forces can be varied with variation of the rotation frequency when a coefficient of variation is common for forces applied to all DOFs. As a result calculations of \( \overline{Q}_{lin} \) for each current value of the rotation speed, \( \omega \), is performed by summation of mode shapes, \( \phi_{r_m} \), containing only nonlinear DOFs, which usually represent a very small fraction of all DOFs in a sector model. In this summation each such mode, \( \phi_{r_m} \), is multiplied by a simple coefficient which is product of the modal force and an explicitly given function of the rotation frequency.

Solution of Nonlinear Equations

Nonlinear Equations Formulated in Real Numbers. One of the most efficient methods for solution of the nonlinear equations is the Newton-Raphson method which possesses quadratic convergence when an approximation is close enough to the solution. An iterative solution process is expressed by the following formula:

\[
\overline{Q}^{(k+1)} = \overline{Q}^{(k)} - \left[ \frac{\partial E(\overline{Q})}{\partial \overline{Q}} \right]^{-1} \left( \overline{E}(\overline{Q}^{(k)}) \right)
\]

(34)

where superscript \( (k) \) indicates the number of the current iteration. Vector of residuals, \( \overline{E}(\overline{Q}) \), can be taken for this equation as from FE and as from FRF formulation. For a case of FE model performing differentiation of Eq. (20) with respect to \( \overline{Q} \) one can obtain the following expression for derivatives of the residuals:

\[
\frac{\partial E(\overline{Q}^{(k)})}{\partial \overline{Q}} = \overline{Z}(\omega) + \frac{\partial E(\overline{Q})}{\partial \overline{Q}} = \overline{Z}(\omega) + \overline{K}_{lin}(\overline{Q})
\]

(35)

where \( \overline{K}_{lin}(\overline{Q}) \) represents a so-called “tangent” stiffness matrix, i.e., a stiffness matrix describing stiffness properties of the nonlinear contact interfaces in vicinity of current values of the displacements, \( \overline{Q} \). An efficient method for analytical derivation which provides explicit, extremely fast and exact expressions of the stiffness matrices and the nonlinear forces at contact interfaces with friction forces, gaps and interferences can be found in papers, [14, 15].

The case of a use of FRF sector matrices can be efficiently analyzed using complex arithmetic and is considered below.

Nonlinear Equations Formulated in Complex Numbers

Although nonlinear equation of motion can be formulated and conveniently evaluated using complex arithmetic but at the stage of their iterative solution by Newton-Raphson method they have to be transformed back into domain of real numbers. This is very important and necessary requirement since residuals, \( \mathcal{E}_j \), are not analytic functions of complex vectors, \( \overline{Q}_j \), and as result of this their derivatives, \( \partial \mathcal{E}_j / \partial \overline{Q}_j \), cannot be defined.
After calculation of complex residuals for a sector, \(\mathbf{E}_j\), which can be made very efficiently using FRF formulation, vector of real residuals is calculated from their complex counterparts as

\[
\mathbf{E} = \{\mathcal{E}_0, \text{Re}(\mathcal{E}_1), \text{Im}(\mathcal{E}_1), \ldots, \text{Im}(\mathcal{E}_n)\}^T.
\]  

(36)

The matrix of derivatives of the residuals involved into Eq. (34) is calculated for the case when complex FRF sector matrices are used by the following way:

\[
\frac{\partial \mathbf{E}^{\text{FRF}}}{\partial \mathbf{Q}} = \mathbf{I} + 
\begin{bmatrix}
\mathbf{D}_{00} & \text{Re}(\mathbf{D}_{01}) & \text{Im}(\mathbf{D}_{01}) & \ldots & \text{Im}(\mathbf{D}_{0n}) \\
\text{Re}(\mathbf{D}_{01}) & \mathbf{D}_{11} & \text{Re}(\mathbf{D}_{12}) & \ldots & \text{Im}(\mathbf{D}_{1n}) \\
\text{Im}(\mathbf{D}_{01}) & \text{Re}(\mathbf{D}_{11}) & \text{Re}(\mathbf{D}_{12}) & \ldots & \text{Im}(\mathbf{D}_{1n}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\text{Im}(\mathbf{D}_{0n}) & \text{Im}(\mathbf{D}_{n1}) & \text{Im}(\mathbf{D}_{n2}) & \ldots & \mathbf{D}_{nn}
\end{bmatrix}
\]  

(37)

where

\[
\mathbf{D}_{ij} = A_{ij} \frac{\partial \mathbf{F}}{\partial \mathbf{Q}_j}, 
\quad \mathbf{D}_{i,j} = A_{ij} \frac{\partial \mathbf{F}}{\partial \mathbf{Q}_j}, 
\quad \mathbf{D}_{i,j+1} = A_{ij} \frac{\partial \mathbf{F}}{\partial \mathbf{Q}_j}
\]

for \(i = 0, n\). As seen in first row of matrix, \(\frac{\partial \mathbf{E}^{\text{FRF}}}{\partial \mathbf{Q}}\), real and imaginary parts of \(\mathbf{D}_{00}\) are alternating and all the other rows contain only real or imaginary parts of \(\mathbf{D}_{ij}\) and these rows are also alternating.

### Numerical Results

#### Comparison of Results Obtained With the Proposed Method and With a Whole Bladed Disk Model

In order to validate the method forced response of cyclically symmetric systems obtained with proposed method was compared with results obtained using whole models of structures when cyclic symmetry properties were disregarded.

As an example of a bladed disk that was calculated using a sector model and a whole bladed disk model, is shown in Fig. 2. The whole bladed disk consists of eight sectors and is modeled by hexahedral finite elements with a total number of degrees-of-freedom in the model equal to 5760; the sector model comprises 720 DOFs. Excitation by first engine order harmonic is considered and the damping loss factor was accepted to be 0.003. Nonlinear interface elements are applied between adjacent blades at a middle node of blade tip faces and as illustrated in Fig. 2 by boxes with arrows.

The nonlinear interface elements produce forces that are dependent on relative displacements of the adjacent blades. Three different types of the nonlinear interface elements were examined: (i) a friction damper (with the friction coefficient 0.3; the normal force \(2 \times 10^3\) N and stiffness coefficient \(4 \times 10^3\) N/mm); (ii) a spring with cubic dependence, \(f(x) = cx^3\), of the nonlinear inter-

action force, \(f(x)\), on relative displacement, \(x\), (stiffness coefficient, \(c\), of the spring is taken to be \(10^3\) N/mm); (iii) a gap element, which does not produce any force unless the prescribed gap is closed and when the gap is closed then interaction force is proportional to the relative displacement (a gap value is taken to be 0.5 mm and stiffness coefficient when the gap is closed is equal to \(10^3\) N/mm). Explicit expressions for tangent stiffness matrix and for multiharmonic components of the interaction forces produced by these elements are used in for calculations in the form as they are analytically derived in papers [14, 15]. For all these interface elements forced response of the bladed disk exhibits strongly nonlinear behavior. First five harmonics (from 0 to 4) are kept in multiharmonic expansion of the forced response and maximum displacements for all three coordinates, \(x\), \(y\), and \(z\), of displacements at the nodes where the nonlinear interface elements were calculated. Results obtained by using the sector model and by using the whole bladed disk are identical. This is demonstrated in Fig. 3, where lines plot results obtained with the whole bladed disk model and results obtained with the proposed method are shown by circles, squares and triangles for \(x\), \(y\), and \(z\) coordinates accordingly.

### Realistic High-Pressure Turbine Bladed Disk

As an example of a practical application a turbine high-pressure bladed disk shown in Fig. 4 is considered. The bladed disk comprises 92 shrouded blades. The damping loss factor is set to 0.003 and excitation by 4, 6, 8, and 16 engine orders is considered in the analysis. Natural frequencies of the high-pressure turbine disk normalized with respect to the first blade-alone frequency are shown in Fig. 5 for all possible nodal diameter numbers from 0 to 46. These frequencies are calculated for the case where there is no contact between shrouds.

Nonlinear forces can occur under certain circumstances during vibration as a result of contact interaction between blade shrouds. These nonlinear forces and nonlinear stiffness matrices of the contact interface were calculated using friction contact interface elements developed in [14]. The contact interface elements allow to take into account unilateral nature of the force normal to the contact surface, friction forces with accounting for the normal load variation and they also allow consistent determination of all stick-slip and contact-separation transitions that occur under the vibrations.

Eleven of the friction interface elements are distributed over nodes of left boundary and eleven interface elements on the right boundary of the sector shroud contacts (locations of the elements on the right sector boundary are shown by circles marked by letter “A” in Fig. 4(c).

Full finite element model of the bladed disk comprises about 15 millions DOFs and the finite single-sector model contains 162,708 degrees-of-freedom (DOFs). Number of DOFs in nonlinear equations is reduced to 66 by applying the developed method for exclusion of linear DOFs from resulting equation while preserving accuracy and all dynamic properties of the initial model since during this reduction there is no any loss of information about influence of the excluded DOFs. In all calculations performed the maximum displacement is determined as

\[
\max \left| x(t) + y(t) + z(t) \right|.
\]

Ratio between amplitudes of all \(r = [0.02z]\) nodes of the bladed disk is varied with excitation frequency and is dependent on the shroud contact conditions but for the bladed disk analyzed the displacements have higher levels at the blade tip in the frequency ranges considered. Because of that the maximum displacement is plotted in all figures for a representative node located at the blade tip. The node is shown by a circle marked by letter “B” in Fig. 4(c).

In Fig. 6 the maximum displacement is shown as a function of excitation frequency for different values of clearances (Fig. 6a) and interferences (Fig. 6b) between shrouds of adjacent sectors. Excitation of engine order type by 4EO is applied to the bladed disk and 4th harmonic component is used in the displacement expansion. For comparison forced responses of linear systems are...
plotted also by thin lines: (i) in Fig. 6(a) forced response of the bladed disk without shroud contacts is plotted and (ii) in Fig. 6(b) forced response of the bladed disk when all nodes are always in full contact without slip is plotted. As seen for the bladed disk with clearances $10^{-2}$ mm and $10^{-3}$ mm the system starts exhibiting strongly nonlinear behavior in vicinities of resonance peaks where vibration displacement are large enough to close the clearance between shrouds. The nonlinear forced response has stiffening characteristics when with increase of amplitude resonance frequency is increasing. Far from resonances the amplitudes are so small that shrouds cannot contact during vibrations and the system has forced responses identical to the linear system. For clearance $10^{-2}$ mm the shrouds come to contact over the whole excitation frequency range. This causes significant reduction of the response level over the analyzed frequency range and even disappearance of one of three resonance peaks that the linear system without shroud contacts has. Moreover, the forced response characteristics takes a more complicated shape, as on can see for frequency range after first resonance and in vicinity of second resonance. For the latter zoomed view is shown to see the curve in more detail. For the bladed disk with interferences shown in Fig. 6(b) one can see that force response exhibits softening characteristics when increase of amplitudes is accompanied by decrease of resonance frequencies.

This happens because increase of amplitude of displacements causes more nodes at the shroud contact surfaces to go out of contact and to slip. For interference values $10^{-4}$, $10^{-5}$, and $10^{-6}$ mm nonlinear effects become evident in the vicinity of resonance and for their values $5 \times 10^{-7}$, $10^{-7}$, and $0$ forced response is different from the linear system compared over the whole frequency range. For these small interference values the amplitudes are much higher that those of the linear system in frequencies ranges far from the resonance whereas they are many times lower.
than those of the linear system at the resonance. This observation is explained by different effects of the shroud nonlinear interaction for excitation frequencies far and close to the resonance. For out-of-resonance regimes the friction damping produced at shroud contact surfaces does not affect significantly level of amplitudes and major effect on the amplitudes is due to decrease of shroud contact stiffness when some or all of the contact nodes lose temporarily contact. For the resonance regime the friction damping is a major cause of the reduction of the amplitude level while change of the stiffness is also reflected in the decrease of the resonance frequency value.

Effects of the clearances and interferences on nonlinear forced response excited by different engine orders is demonstrated in Fig. 7 where cases of the clearance value $10^{-2}$ mm (Fig. 7(a)) and the interference value $10^{-6}$ mm (Fig. 7(b)) are analyzed. Forced responses of the corresponding linear systems are also plotted for comparison by thin lines. For the case of the system with the clearances one can see that the higher engine order number, the lower amplitude level when shroud contact occurs.

This is explained by the fact that to close the gap between adjacent shrouds their relative displacements have to be large enough. Amplitudes are the same for adjacent blades of the cyclically symmetric bladed disk but a phase shift between the dominant harmonics of the displacements is proportional to the excitation engine order number. For larger engine order numbers due to larger phase shift the relative displacements can be larger even when the absolute blade amplitudes are smaller. For both cases of the considered clearances and the interferences the friction damping is significant although for the case with the interferences its effect much more evident.

Conclusions

An effective method for analysis of periodic forced response of nonlinear cyclically symmetric structures has been developed. A rigorous proof of the validity of the reduction of the whole nonlinear structure to a sector is provided for a general case of nonlinear forces of arbitrary character and nature. Types of bladed disk forcing permitting application of the method are discussed. The method allows multiharmonic nonlinear forced response for a whole bladed disk to be calculated using a sector model without any loss of accuracy in calculations and modeling. A multiharmonic formulation for equations of motion has been derived for two widely used techniques of description of the linear part of a bladed disk model with the large number of degrees-of-freedom: (i) using sector finite element matrices; (ii) using sector natural mode shapes and frequencies. An approach for a use of complex arithmetic in the multiharmonic formulation and in the solution of the nonlinear equations of motion has been examined. A technique for reduction of size of the nonlinear multiharmonic equations by exclusion of linear DOFs from the resulting equations has been developed.
The numerical investigations show high efficiency of the proposed method. The identity of the results obtained with the developed method using a sector model and with a whole bladed disk model has been demonstrated for different types of nonlinear forces produced by nonlinear interfaces: (i) friction dampers; (ii) gaps; (iii) cubic nonlinear springs. A practical, large finite element model of a shrouded turbine bladed disk has been analyzed with accounting for friction forces, clearances, and interferences at nodes located at surfaces of shroud contacts.

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